Internal Categories and Quantum Groups

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

Marcelo Aguiar

August 1997

Internal Categories and Quantum Groups

Marcelo Aguiar, Ph.D.

Cornell University 1997

Let $\mathcal{S}$ be a monoidal category with equalizers that are preserved by the tensor product. The notion of *categories internal to* $\mathcal{S}$ is defined, generalizing the notions of monoid and comonoid in $\mathcal{S}$, and extending the usual notion of internal categories, which is obtained when $\mathcal{S}$ is a category with products and equalizers.

The basic theory of internal categories is developed and several applications to quantum groups are found. Deltacategories are defined; these are algebraic objects that generalize groups or bialgebras, in the sense that attached to them there is a monoidal category of representations. Quantum groups are constructed from deltacategories. In particular a contruction of quantum groups generalizing that of Drinfeld and Jimbo is presented. An invariant of finite dimensional quasitriangular Hopf algebras is constructed.

BIOGRAPHICAL SKETCH

Marcelo Aguiar-Solari was born on October 7, 1968 to his parents Hugo and Alicia, and two older brothers Alvaro and Gonzalo, in Montevideo, Uruguay. He attended elementary and high school at the Escuela y Liceo J. Zorrilla, Hnos. Maristas, in Punta Carretas, Montevideo. The step that would inevitably take him into Mathematics was taken in 1987, when he matriculated at the Facultad de Ciencias of the Universidad de Uruguay. He graduated from that institution in 1991, and following the advice of his advisor Walter Ferrer-Santos, seeked to pursue graduate studies in the United States. Life had him end up in Ithaca, a marvelous place where he would get to make the best friends, learn to dance Merengue and dare to try Salsa and, perhaps a bit paradoxically, play more soccer than ever before. After five years of serious but enjoyable work, not only has he earned a Ph.D. in Mathematics, but also won the Cornell Soccer Intramurals (Spring 96) and the City of Ithaca Soccer League (Spring 97) and most importantly, the friendship of many, without which no other achievements would have been possible.

Marcelo has been a Peñarol loyal fan for over 28 years now, and plans to remain so forever. Love for this greatest team, the sea, his family and old friends, have had Marcelo return to Uruguay more often than not.

Marcelo is a little sad to think that soon he will be leaving Ithaca, its gorges, fields and friends.

To my parents, with love and gratitude.

ACKNOWLEDGEMENTS

I owe a great deal to many, and this is only a partial account of my indebtedness.

This thesis would not have been possible without the guidance and support of Steve Chase. I thank him for many hours spent with me, teaching me, and listening to and criticizing my ideas. His many careful readings of drafts of this thesis, and the resulting suggestions, were invaluable.

I thank others at Cornell, for teaching me and listening to my developing work; specially, Professors Dan Barbasch, Ken Brown, Richard Erhenborg, Allen Hatcher, Peter Kahn, Tom Rishel and Moss Sweedler.

I thank Walter Ferrer, from the University of Uruguay, for his initial advice and constant support, and his friendship throughout these years.

My gratitude extends to the University of Uruguay, for providing me the opportunity to initiate me in Mathematics, and for its support, and also to Cornell University, for opening to me the doors to a fantastic place and offering me excellent scientific education.

I thank my friends, from Uruguay and Ithaca, for their most valuable gift. Among them, Andres and Carmen, Ricardo, Marta and Manuel, Gonzalo and Silvana, Patricia, Francesca, Reed, Suman, Marcelo, Leo, Fernando, Juan Pablo, Maca,

Coco, Fer.

My deepest gratitude goes to my family, for their years of love and support. My Dad and Mom, my brothers, and my favorites, nephews Luis and Diego and niece Alicia -I love you all.

Table of Contents

List of Tables

Part I

Internal Category Theory

In this part we develop the basic theory of internal categories and their morphisms. The main goals are to introduce the notions of admissible sections of an internal category and that of deltacategories.

In chapter 1 we summarize the basic results needed. Although this makes the presentation considerably self-contained, familiarity with the basic notions of category theory, in particular monoidal categories, is probably a prerequisite.

Throughout, a fixed monoidal category $\mathcal{S}$ will underlie all constructions. Categories internal to $\mathcal{S}$ are defined in section 2.3. The usual notion of internal categories [Joh 2.1] is obtained when $\mathcal{S}$ is a category with products and equalizers; from the point of view of this work, though, the interesting cases arise when considering more general monoidal categories.

Monoids and comonoids in $\mathcal{S}$ are particular examples of categories internal to $\mathcal{S}$, but these are in a way the trivial examples. Other basic examples will be discussed in section 2.4 and as the theory of internal categories is developed in the subsequent sections. The most interesting examples and applications are postponed for later parts.

Different choices of $\mathcal{S}$ yield different types of internal categories, some of which have been considered in the literature under various names, e.g. linear categories [Mit], algebroids (called graphs in [Mal]), coalgebroids [Del], bialgebroids [Rav, Mal] and others; see table 2.1 in section 2.4.

Most constructions on internal categories that we introduce (admissible sections, representations) are functorial with respect to functors that are the identity on objects, but not with respect to arbitrary functors. Notably, there is an alternative notion of morphism, called cofunctors, with respect to which these constructions are functorial. Functors and cofunctors, and the corresponding "natural morphisms" among them are discussed in sections 4.1 and 4.2.

The monoid of admissible sections is an ordinary monoid in Sets that is attached to every internal category (chapter 5). This assignment generalizes many constructions already in the literature for the case of ordinary categories.

Each internal category has a category of representations (chapter 6). This is an ordinary category consisting of objects of $\mathcal{S}$ where the internal category acts in some sense.

A deltacategory (section 7.4) is an internal category for which its category of representations is monoidal in a natural way. These are interesting from the point of view of quantum group theory, since the monoid of admissible sections of a deltacategory in Vec_k carries a structure of k-bialgebra. This particular application of the general theory to quantum groups will be developed in chapter 9.

Deltacategories in Sets appear to be very interesting objects as well; several examples of these are worked out in chapter 8. Categories in Sets are just usual small categories; one may think of them as "monoids with several objects". Deltacategories match this intuition even better, since, as monoids, they yield bialgebras (after linearization) and monoidal categories of representations, as outlined in table 8.1.

Chapter 1

Preliminaries on category theory

In this section we collect various basic results in category theory that will get used throughout this work.

1.1 Forks and equalizers

A *fork* in a category $\mathcal{S}$ is a diagram of the form

$$\begin{tikzcd} E \arrow[r, "i"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd} \text{ where } fi = gi \ .$$

The arrow i is an *equalizer* of the pair f, g if $fi = gi$ and for any $j : E' \to X$ with $fj = gj$, there is a unique $e : E' \to E$ such that $j = ei$.

$$\begin{tikzcd} E \arrow[r, "i"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ E' \arrow[u, dashed, "e"] \arrow[ur, "j"'] & & \end{tikzcd}$$

In this case i is unique up to isomorphism; we sometimes write $E = \mathsf{Eq}_{\mathcal{S}}(f, g)$ and refer to i as the *canonical map* $E \xrightarrow{\mathsf{can}} X$. This map is always a monomorphism.

A *split fork* is a fork as above with two more arrows $h : Y \to X$ and $p : X \to E$ such that

$$hf = \mathsf{id}_X,\ hg = ip \text{ and } pi = \mathsf{id}_E\ .$$

Lemma 1.1.1. *In every split fork, i is the equalizer of f and g.*

Proof. This is lemma VI.6 from [ML], in dual from. (Mac Lane uses "fork" for what we would call "coforks"). ☐

1.2 Coreflexive pairs

A *coreflexive pair* in a category $\mathcal{S}$ is a pair $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ for which there exists an arrow $h : Y \to X$ such that $hf = hg = \mathsf{id}_X$.

The following useful result will be referred to as Johnstone's lemma.

Lemma 1.2.1. *Let*

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"'] & X_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[d, "\beta_0"'] & X_2 \arrow[d, "\gamma_0"] \\
Y_0 \arrow[r, "g_0"] \arrow[d, shift right, "\alpha_2"'] \arrow[d, shift left, "\alpha_1"] & Y_1 \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[d, shift right, "\beta_2"'] \arrow[d, shift left, "\beta_1"] & Y_2 \arrow[d, shift right, "\gamma_2"'] \arrow[d, shift left, "\gamma_1"] \\
Z_0 \arrow[r, "h_0"'] & Z_1 \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & Z_2
\end{tikzcd}$$

be a diagram in a category $\mathcal{S}$ such that

- *the squares commute in the "obvious" way, that is*

$$g_0\alpha_0 = \beta_0 f_0$$

$$h_0\alpha_i = \beta_i g_0,\ \ i = 1, 2$$

$$g_i\beta_0 = \gamma_0 f_i,\ \ i = 1, 2$$

$$h_i\beta_j = \gamma_j g_i,\ \ 1 \le i, j \le 2$$

- *the rows and columns are equalizer diagrams*

- *the pairs* (γ_1, γ_2) *and* (h_1, h_2) *are coreflexive.*

Then the diagonal

$$\begin{tikzcd} X_0 \arrow[r, "g_0\alpha_0"] & Y_1 \arrow[r, shift left, "h_1\beta_1"] \arrow[r, shift right, "h_2\beta_2"'] & Z_2 \end{tikzcd}$$

is an equalizer.

Proof. This is lemma 0.17 in [Joh], in dual form. ☐

The hypothesis of Johnstone's lemma can be slightly weakened, in view of the following.

Lemma 1.2.2. *Let*

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"'] & X_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[d, "\beta_0"'] & X_2 \arrow[d, "\gamma_0"] \\ Y_0 \arrow[r, "g_0"] \arrow[d, shift right, "\alpha_2"'] \arrow[d, shift left, "\alpha_1"] & Y_1 \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[d, shift right, "\beta_2"'] \arrow[d, shift left, "\beta_1"] & Y_2 \arrow[d, shift right, "\gamma_2"'] \arrow[d, shift left, "\gamma_1"] \\ Z_0 \arrow[r, "h_0"'] & Z_1 \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & Z_2 \end{tikzcd}$$

be a diagram in a category $\mathcal{S}$ *where all squares commute as before and all three rows and columns* 1 *and* 2 *are equalizers. Then column* 0 *is also an equalizer.*

Proof. First,

$$h_0\alpha_2\alpha_0 = \beta_2 g_0\alpha_0 = \beta_2\beta_0 f_0 = \beta_1\beta_0 f_0 = \beta_1 g_0\alpha_0 = h_0\alpha_1\alpha_0;$$

hence, $\alpha_2\alpha_0 = \alpha_1\alpha_0$, since h_0 is monic.

Now let $E \xrightarrow{\alpha} Y_0$ be such that $\alpha_2\alpha = \alpha_1\alpha$. We have to show that α factors through α_0 to complete the proof (uniqueness of the factorization follows from the fact that α_0 is monic, which holds since so is $g_0\alpha_0 = \beta_0 f_0$). To this end, consider the following diagram.

$$\begin{tikzcd}
E \arrow[r, dashed, "\tilde{\alpha}"] \arrow[rr, dashed, bend left, "\tilde{\alpha_1}"] \arrow[dr, "\alpha"'] & X_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"'] & X_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[d, "\beta_0"'] & X_2 \arrow[d, "\gamma_0"] \\
& Y_0 \arrow[r, "g_0"] \arrow[d, shift right, "\alpha_2"'] \arrow[d, shift left, "\alpha_1"] & Y_1 \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[d, shift right, "\beta_2"'] \arrow[d, shift left, "\beta_1"] & Y_2 \arrow[d, shift right, "\gamma_2"'] \arrow[d, shift left, "\gamma_1"] \\
& Z_0 \arrow[r, "h_0"'] & Z_1 \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & Z_2
\end{tikzcd}$$

First we show that $g_0\alpha$ factors through β_0, using that column 1 is an equalizer. Indeed,

$$\beta_2 g_0\alpha = h_0\alpha_2\alpha = h_0\alpha_1\alpha = \beta_1 g_0\alpha,$$

so an arrow $\tilde{\alpha_1} : E \to X_1$ such that

$$\beta_0\tilde{\alpha_1} = g_0\alpha \tag{*}$$

exists. Now,

$$\gamma_0 f_2\tilde{\alpha_1} = g_2\beta_0\tilde{\alpha_1} \overset{(*)}{=} g_2 g_0\alpha = g_1 g_0\alpha \overset{(*)}{=} g_1\beta_0\tilde{\alpha_1} = \gamma_0 f_1\tilde{\alpha_1};$$

hence, $f_2\tilde{\alpha}_1 = f_1\tilde{\alpha}_1$, since γ_0 is monic. Thus, since row X is an equalizer, there is an arrow $\tilde{\alpha} : E \to X_0$ such that

$$f_0\tilde{\alpha} = \tilde{\alpha}_1 \tag{**}$$

Hence

$$g_0\alpha \overset{(*)}{=} \beta_0\tilde{\alpha}_1 \overset{(**)}{=} \beta_0 f_0\tilde{\alpha} = g_0\alpha_0\tilde{\alpha},$$

from where, since g_0 is monic,

$$\alpha = \alpha_0\tilde{\alpha}$$

as needed. ☐

Remark 1.2.1. The above result is also a direct consequence of the result on page 227 of [ML] on iterated limits. Take $\mathfrak{C} = 1 \rightrightarrows 2$ and $F : \mathfrak{C} \times \mathfrak{C} \to \mathcal{S}$, $F(1,1) = Y_1$, $F(1,2) = Y_2$, $F(2,1) = Z_1$ and $F(2,2) = Z_2$. Then, by assumption,

$$\lim F(1,-) = Y_0,\ \lim F(2,-) = Z_0,\ \lim F(-,1) = X_1 \text{ and } \lim F(-,2) = X_2,$$

from where

$$\lim(\ Y_0 \rightrightarrows Z_0\ ) = \lim_i \lim_j F(i,j) = \lim_j \lim_i F(i,j) = \lim(\ X_1 \rightrightarrows X_2\ ) = X_0\ ,$$

which is the desired conclusion. The same type of argument can be used for proving lemma 1.2.1.

1.3 2-categories

For the basic material on (strict) 2-categories the reader is referred to [KS].

A *strict* 2-*category* $\mathcal{K}$ has objects or 0-cells $A, B, \dots$, arrows or 1-cells $f, g, \dots$ and 2-cells $\alpha, \beta, \dots$, linked by means of source and target maps as suggested by the following picture:

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] & B \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd} .$$

These data should satisfy the following conditions:

- The objects and the arrows form an ordinary category, with identities id_A and composition depicted as
  $$A \xrightarrow{f} B \xrightarrow{g} C = A \xrightarrow{gf} C$$
- For each pair of objects A, B, the arrows $A \to B$ and the 2-cells between them form a category, with identities id_f and composition depicted as
  $$\begin{tikzcd} A \arrow[r, bend left=60, "f"{name=U}] \arrow[r, "g"{name=M, description}] \arrow[r, bend right=60, "h"'{name=D}] & B \arrow[Rightarrow, from=U, to=M, "\alpha"] \arrow[Rightarrow, from=M, to=D, "\beta"] \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "h"'{name=D}] & B \arrow[Rightarrow, from=U, to=D, "\beta\circ\alpha"] \end{tikzcd}$$
  These compositions and identities are called the *vertical structure* of $\mathcal{K}$.
- The objects and the 2-cells form a category, with identities $\mathsf{id}_{\mathsf{d}_A}$ and composition depicted as
  $$\begin{tikzcd} A \arrow[r, bend left=50, "f"{name=U1}] \arrow[r, bend right=50, "g"'{name=D1}] & B \arrow[r, bend left=50, "u"{name=U2}] \arrow[r, bend right=50, "v"'{name=D2}] & C \arrow[Rightarrow, from=U1, to=D1, "\alpha"] \arrow[Rightarrow, from=U2, to=D2, "\beta"] \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left=50, "uf"{name=U}] \arrow[r, bend right=50, "vg"'{name=D}] & C \arrow[Rightarrow, from=U, to=D, "\beta*\alpha"] \end{tikzcd}$$
  These composition and identities are called the *horizontal structure* of $\mathcal{K}$.

- The vertical and horizontal category structures are compatible in the sense that, in the situation

  $$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U1}] \arrow[r, bend right, "f"'{name=D1}] & B \arrow[r, bend left, "u"{name=U2}] \arrow[r, bend right, "u"'{name=D2}] & C \arrow[from=U1, to=D1, Rightarrow, "\mathsf{id}_f"] \arrow[from=U2, to=D2, Rightarrow, "\mathsf{id}_u"] \end{tikzcd}$$

  we have

  $$\mathsf{id}_u * \mathsf{id}_f = \mathsf{id}_{uf} \ ,$$

  and in the situation

  $$\begin{tikzcd}[column sep=huge] A \arrow[r, bend left=60, "f"{name=F}] \arrow[r, "g"{name=G, description}] \arrow[r, bend right=60, "h"'{name=H}] & B \arrow[r, bend left=60, "u"{name=U}] \arrow[r, "v"{name=V, description}] \arrow[r, bend right=60, "w"'{name=W}] & C \arrow[from=F, to=G, Rightarrow, "\alpha"] \arrow[from=G, to=H, Rightarrow, "\beta"] \arrow[from=U, to=V, Rightarrow, "\gamma"] \arrow[from=V, to=W, Rightarrow, "\delta"] \end{tikzcd}$$

  we have

  $$(\delta \circ \gamma) * (\beta \circ \alpha) = (\delta * \beta) \circ (\gamma * \alpha) \ .$$

The prototypical example of a 2-category is $\mathcal{K} = \boldsymbol{LCat}$, where the objects are (large) categories, the arrows are functors and the 2-cells are natural transformations.

Another example is $\mathcal{K} = \boldsymbol{ALG}_k$, where the objects are k-algebras (k being a fixed field), an arrow $A \xrightarrow{M} B$ is a B-A-bimodule M, and the 2-cells are the morphisms of bimodules. Here vertical composition is the usual composition of maps, while horizontal composition is the tensor product of bimodules:

$$\begin{tikzcd} A \arrow[r, bend left, "M"{name=M}] \arrow[r, bend right, "N"'{name=N}] & B \arrow[r, bend left, "P"{name=P}] \arrow[r, bend right, "Q"'{name=Q}] & C \arrow[from=M, to=N, Rightarrow, "\alpha"] \arrow[from=P, to=Q, Rightarrow, "\beta"] \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "P\otimes_B M"{name=PM}] \arrow[r, bend right, "Q\otimes_B N"'{name=QN}] & C \arrow[from=PM, to=QN, Rightarrow, "\beta\otimes_B\alpha"] \end{tikzcd} \ .$$

Other examples will be dealt with in later sections.

1.4 Monoidal categories

The main references for this section are [K, chapter XI] and [P1].

A *strict monoidal category* $\mathcal{S}$ is a (strict) 2-category with just one object. It is easy to see that this structure can be described more directly as follows. $\mathcal{S}$ is an ordinary category with objects $A, B, \dots$, arrows $f, g, \dots$, and in addition there is given a functor $\otimes : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ and an object I such that

- $f \otimes (g \otimes h) = (f \otimes g) \otimes h$ for all arrows f, g, h, in particular $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ for all objects A, B, C, and
- $\mathsf{id}_I \otimes f = f = f \otimes \mathsf{id}_I$ for all arrows f, in particular $I \otimes A = A = A \otimes I$ for all objects A.

In practice we will meet *non-strict* monoidal categories, which we will treat as if they were strict. This is legitimate in view of theorem XI.5.3 in [K].

The category $\mathcal{S} = \mathit{Sets}$ of all sets is monoidal, taking $\otimes$ to be just the cartesian product and I a one-element set. More generally, any category with finite products and a final object can be seen as a monoidal category in this way [ML, proposition III.5]. Such categories are called *lex* categories.

The category $\mathcal{S} = \mathit{Vec}_k$ of all vector spaces over a field k is monoidal, taking $\otimes = \otimes_k$ to be the usual tensor product over k, and $I = k$ a one-dimensional space. Several other monoidal categories will be considered in this work.

Notice that if $\mathcal{K}$ is a 2-category, then for any object A of $\mathcal{K}$, there is a monoidal category $\mathcal{S} = \mathcal{K}(A, A)$ whose objects are the arrows $A \to A$ of $\mathcal{K}$ and whose morphisms are the 2-cells of $\mathcal{K}$ among these arrows. The tensor product in $\mathcal{S}$ comes from the horizontal composition in $\mathcal{K}$. In section 2.2 we will define a certain 2-category $\mathcal{G}$, then internal categories will be defined as monoids in the monoidal category $\mathcal{G}(C, C)$.

A *monoid* in a (strict) monoidal category $\mathcal{S}$ is a triple (A, μ, u) where A is an object of $\mathcal{S}$ and $\mu : A \otimes A \to A$ and $u : I \to A$ are morphisms such that both diagrams below commute:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu \otimes \mathsf{id}_A"] \arrow[d, "\mathsf{id}_A \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes A \arrow[r, "u \otimes \mathsf{id}_A"] \arrow[dr, equal] & A \otimes A \arrow[d, "\mu"] & A \otimes I \arrow[l, "\mathsf{id}_A \otimes u"'] \arrow[dl, equal] \\
& A &
\end{tikzcd}$$

A *module* over a monoid A (or an A-module) is a pair (M, χ) where M is an object of $\mathcal{S}$ and $\chi : A \otimes M \to M$ is a morphism such that both diagrams below commute:

$$\begin{tikzcd}
A \otimes A \otimes M \arrow[r, "\mu \otimes \mathsf{id}_M"] \arrow[d, "\mathsf{id}_A \otimes \chi"'] & A \otimes M \arrow[d, "\chi"] \\
A \otimes M \arrow[r, "\chi"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes M \arrow[r, "u \otimes \mathsf{id}_M"] \arrow[dr, equal] & A \otimes M \arrow[d, "\chi"] \\
& M
\end{tikzcd}$$

Let A and B be two monoids in $\mathcal{S}$. A morphism of monoids from A to B is a morphism $f : A \to B$ in $\mathcal{S}$ such that both diagrams below commute:

$$\begin{tikzcd}
A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\mu_A"'] & B \otimes B \arrow[d, "\mu_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "f"] & & B \\
& I \arrow[ul, "\mathsf{u}_A"] \arrow[ur, "\mathsf{u}_B"'] &
\end{tikzcd}$$

The category of all monoids in $\mathcal{S}$ is denoted by $\mathit{Mon}_{\mathcal{S}}$.

Let M and N be two A-modules. A morphism of A-modules from M to N is a morphism $f : M \to N$ in $\mathcal{S}$ such that the diagram below commutes:

$$\begin{tikzcd}
A \otimes M \arrow[r, "\mathsf{id}_A \otimes f"] \arrow[d, "\chi_M"'] & A \otimes N \arrow[d, "\chi_N"] \\
M \arrow[r, "f"'] & N
\end{tikzcd}$$

The category of all A-modules is denoted by $\mathsf{Mod}_{\mathcal{S}} A$.

Comonoids and *comodules* in $\mathcal{S}$ can be defined similarly, by reversing all arrows in the above diagrams. We use the notation (C, Δ, ϵ) for comonoids and (M, t) for C-comodules. The resulting categories are denoted $\boldsymbol{\mathsf{Comon}}_{\mathcal{S}}$ and $\mathsf{Comod}_{\mathcal{S}} C$, respectively. Equivalently, one may define

$$\boldsymbol{\mathsf{Comon}}_{\mathcal{S}} = (\boldsymbol{\mathsf{Mon}}_{\mathcal{S}^{op}})^{op} \text{ and } \mathsf{Comod}_{\mathcal{S}} C = (\mathsf{Mod}_{\mathcal{S}^{op}} C)^{op} \ ,$$

where $\mathcal{K}^{op}$ denotes the category obtained from a category $\mathcal{K}$ by reversing all the arrows (if $\mathcal{K}$ is monoidal then so is $\mathcal{K}^{op}$, under the same tensor product).

Since comonoids and comodules are the building blocks of the notion of internal categories, and we will deal with them throughout this work, it is worth displaying the conditions in their definitions explicitly.

For a comonoid (C, Δ, ϵ), the following commutative diagrams are known as coassociativity and counitality respectively:

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathsf{id}_C \otimes \Delta"] \\
C \otimes C \arrow[r, "\Delta \otimes \mathsf{id}_C"'] & C \otimes C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes C & C \otimes C \arrow[l, "\epsilon \otimes \mathsf{id}_C"'] \arrow[r, "\mathsf{id}_C \otimes \epsilon"] & C \otimes I \\
& C \arrow[u, "\Delta"] \arrow[ul, equal] \arrow[ur, equal] &
\end{tikzcd}$$

Similarly, for a C-comodule (M, t), the following commutative diagrams are known as coassociativity and counitality respectively:

$$\begin{tikzcd}
M \arrow[r, "t"] \arrow[d, "t"'] & C\otimes M \arrow[d, "\mathsf{id}_C\otimes t"] \\
C\otimes M \arrow[r, "\Delta\otimes t"'] & C\otimes C\otimes M
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes M \arrow[dr, equal] & C\otimes M \arrow[l, "\epsilon\otimes\mathsf{id}_M"'] \\
& M \arrow[u, "\chi"']
\end{tikzcd}$$

The above are the definitions for *left* A-modules and C-comodules. The right and mixed versions are defined in the obvious way. For instance, a C-D-bicomodule is a triple (M, s, t) where (M, t) is a left C-comodule, (M, s) is a right D-comodule, and the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "t"] \arrow[d, "s"'] & C\otimes M \arrow[d, "\mathsf{id}_C\otimes s"] \\
M\otimes D \arrow[r, "t\otimes\mathsf{id}_D"'] & C\otimes M\otimes D
\end{tikzcd} .$$

The category of C-D-bicomodules in $\mathcal{S}$ is denoted by $\mathsf{Bicomod}_{\mathcal{S}}(C, D)$.

Suppose now that $\mathcal{S}$ is a *symmetric* monoidal category. This means that for every pair of objects A and B of $\mathcal{S}$ there is a natural isomorphism $\tau_{A,B} : A\otimes B \to B\otimes A$ satisfying some conditions [K, XIII.1.5]. In this case, both $\mathsf{Mon}_{\mathcal{S}}$ and $\mathsf{Comon}_{\mathcal{S}}$ are monoidal categories under the tensor product of $\mathcal{S}$. More precisely, if A and B are monoids in $\mathcal{S}$ then so is $A\otimes B$, via

$$(A\otimes B)\otimes(A\otimes B) \xrightarrow{\mathsf{id}_A\otimes\tau_{B,A}\otimes\mathsf{id}_B} A\otimes A\otimes B\otimes B \xrightarrow{\mu_A\otimes\mu_B} A\otimes B \text{ and } I = I\otimes I \xrightarrow{\mathsf{u}_A\otimes\mathsf{u}_B} A\otimes B \ ;$$

and dually for comonoids. Moreover, $\mathsf{Mon}_{\mathcal{S}}$ and $\mathsf{Comon}_{\mathcal{S}}$ inherit the symmetric structure from $\mathcal{S}$ as well.

Under this assumption on $\mathcal{S}$, a *bimonoid* in $\mathcal{S}$ may be defined equivalently as a comonoid in $\mathsf{Mon}_{\mathcal{S}}$ or as a monoid in $\mathsf{Comon}_{\mathcal{S}}$. Explicitly, a bimonoid in $\mathcal{S}$ is a 5-tuple

$(H, \mu, u, \Delta, \epsilon)$ such that (H, μ, u) is a monoid in $\mathcal{S}$, (H, Δ, ϵ) is a comonoid in $\mathcal{S}$ and the following diagrams commute:

$$\begin{tikzcd}
 & H \arrow[dr, "\Delta"] & \\
H \otimes H \arrow[ur, "\mu"] \arrow[d, "\Delta \otimes \Delta"'] & & H \otimes H \\
H \otimes H \otimes H \otimes H \arrow[rr, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"'] & & H \otimes H \otimes H \otimes H \arrow[u, "\mu \otimes \mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & H \arrow[dr, "\epsilon"] & \\
I \arrow[ur, "u"] \arrow[rr, equal] & & I
\end{tikzcd}$$

$$\begin{tikzcd}
H \otimes H \arrow[r, "\epsilon \otimes \epsilon"] \arrow[d, "\mu"'] & I \otimes I \arrow[d, equal] \\
H \arrow[r, "\epsilon"'] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "u"] \arrow[d, equal] & H \arrow[d, "\Delta"] \\
I \otimes I \arrow[r, "u \otimes u"'] & H \otimes H
\end{tikzcd}$$

The category of bimonoids is $\mathcal{S}$ is $\mathtt{Bimon}_{\mathcal{S}} = \mathtt{Comon}_{\mathtt{Mon}_{\mathcal{S}}} = \mathtt{Mon}_{\mathtt{Comon}_{\mathcal{S}}}$.

Examples 1.4.1.

1. When $\mathcal{S} = \mathtt{Vec}_k$, monoids, comonoids and bimonoids are respectively k-algebras, k-coalgebras and k-bialgebras.

2. When $\mathcal{S} = \mathtt{Sets}$, we get the usual definition of monoids. Every set X carries a unique structure of comonoid, given by the diagonal map $\Delta_X : X \to X \times X$, $\Delta_X(x) = (x, x)$, and $\epsilon_X : X \to I$, $\epsilon_X(x) = *$, where $I = \{*\}$. This is an obvious consequence of coassociativity and counitality. Hence bimonoids coincide with monoids in this case. Moreover, X-comodules admit a simple description as well. Namely, any (left) comodule structure map $t : M \to X \times M$ is necessarly of the form $t(m) = (\tilde{t}(m), m)$ where $\tilde{t} : M \to X$ is some (arbitrary) map. Thus

we sometimes refer to X-comodules as X-graded sets, and to $\tilde{t}$ as the degree map.

3. The same situation arises more generally when $\mathcal{S}$ is any lex category. Any object X of $\mathcal{S}$ admits a unique comonoid structure, given by $\Delta_X = (\mathsf{id}_X, \mathsf{id}_X)$ and the unique map ϵ_X from X to the final object I. Similarly, X-comodules are just X-graded objects of $\mathcal{S}$. These observations are rather trivial, but they provide the key for finding the right generalization of the notion of internal categories from lex categories to more general monoidal categories, as we shall see (section 2.3).

Chapter 2

Definition and basic examples

In this chapter internal categories are defined. This is the main concept of this work. Some basic examples will be discussed as the basic theory of internal categories is developed in later chapters of this part. The main examples and applications are postponed until chapters 9 and 10.

2.1 Regular monoidal categories

Let $\mathcal{S}$ be a monoidal category in which every pair of parallel arrows has an equalizer. Let $A \overset{f}{\underset{g}{\rightrightarrows}} A'$ be a parallel pair of arrows, and B and C objects of $\mathcal{S}$. Let $\mathsf{can} : \mathsf{Eq}_{\mathcal{S}}(A \overset{f}{\underset{g}{\rightrightarrows}} A') \to A$ be the canonical map. There is always a canonical map as follows

$$B\otimes\mathsf{Eq}_{\mathcal{S}}(A \overset{f}{\underset{g}{\rightrightarrows}} A')\otimes C \xrightarrow{\mathsf{id}_B\mathsf{can}\otimes\mathsf{id}_C} \mathsf{Eq}_{\mathcal{S}}(B\otimes A\otimes C \overset{\mathsf{id}_B\otimes f\otimes\mathsf{id}_C}{\underset{\mathsf{id}_B\otimes g\otimes\mathsf{id}_C}{\rightrightarrows}} B\otimes A'\otimes C) \ , \qquad (*)$$

since $(\mathsf{id}_B\otimes f\otimes\mathsf{id}_C)\circ(\mathsf{id}_B\otimes\mathsf{can}\otimes\mathsf{id}_C) = (\mathsf{id}_B\otimes g\otimes\mathsf{id}_C)\circ(\mathsf{id}_B\otimes\mathsf{can}\otimes\mathsf{id}_C)$.

Definition 2.1.1. A monoidal category $\mathcal{S}$ is called *regular* if it possesses equalizers for every pair of parallel arrows and, furthermore, the canonical map (*) is always an isomorphism.

We say that equalizers are preserved by the tensor product in this case.

Examples 2.1.1.

1. The monoidal category Sets is regular: if (b, a, c) belongs to $\mathsf{Eq}_{\mathcal{S}}(\mathsf{id}_B{\otimes}f{\otimes}\mathsf{id}_C,$ $\mathsf{id}_B{\otimes}g{\otimes}\mathsf{id}_C)$, then $f(a) = g(a)$, and hence (b, a, c) belongs to $B{\otimes}\mathsf{Eq}_{\mathcal{S}}(f, g){\otimes}C$.

   More generally, any lex category is regular. In fact, in view of Yoneda's lemma [ML,III.2], one can argue by using the same proof as for Sets, where now b, a and c are to be interpreted as "generalized elements".

2. The monoidal category Vec_k is regular. This follows from the fact that for any pair of linear maps $f : V \to V'$ and $g : W \to W'$, $\mathsf{Ker}(f{\otimes}g) = \mathsf{Ker}f{\otimes}W + V{\otimes}\mathsf{Ker}g$.

   More generally, if R is a commutative ring and $\mathcal{S} = \mathsf{Mod}_R$ viewed as a monoidal category under $\otimes_R$, then $\mathcal{S}$ is regular if and only if all R-modules are flat. (This condition is equivalent to R being a regular ring in the sense of von Neumann by [Row, proposition 2.11.20]; this is the reason for the chosen terminology). In fact, if $\mathcal{S}$ is regular, then equalizers and in particular kernels and monomorphisms are preserved by $\otimes_R$, hence R is regular. Conversely, if left exact sequences are preserved, consideration of the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{Eq}_{\mathcal{S}}(A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A') \arrow[r] & A \arrow[r, "f-g"] & A' \end{tikzcd}$$

shows that $\mathsf{Eq}_{\mathcal{S}}(A \overset{f}{\underset{g}{\rightrightarrows}} A')$ is preserved under $\otimes_R$.

3. It follows from the above examples plus proposition 2.1.1 below that $\mathcal{M}onoids$, the category of ordinary *set*-monoids, and $\mathcal{A}lg_k$, the category of k-algebras, are regular monoidal categories. So is $\mathcal{G}roups$.

4. Let $\mathcal{S}$ be a monoidal category. Then $\mathcal{S}^{op}$ (a monoidal category under the same tensor product as that of $\mathcal{S}$) is regular if and only if the tensor product preserves *coequalizers* in $\mathcal{S}$. $\mathcal{V}ec_k^{op}$, and more generally Mod_R^{op} for any ring R, is regular, since $\otimes_R$ is always right exact. $\mathcal{S}ets^{op}$ is also regular, by [ML, exercise V.4.4].

5. $\mathcal{A}lg_k^{op}$ is regular. In fact, the coequalizer in $\mathcal{A}lg_k$ of $A \overset{f}{\underset{g}{\rightrightarrows}} A'$ is $A'/I(f,g)$, where $I(f,g)$ is the ideal generated by $\{f(a) - g(a) \ / \ a \in A\}$. Now, if B is any other k-algebra, clearly

$$I(\mathsf{id}_B \otimes f, \mathsf{id}_B \otimes g) = \langle \{b \otimes f(a) - b \otimes g(a) \ / \ b \in B, a \in A\} \rangle = B \otimes I(f,g) \ ;$$

hence, $\mathsf{Coeq}_{\mathcal{A}lg_k}(\mathsf{id}_B \otimes f, \mathsf{id}_B \otimes g) =$

$$= B \otimes A'/I(\mathsf{id}_B \otimes f, \mathsf{id}_B \otimes g) = B \otimes A'/B \otimes I(f,g) = B \otimes \mathsf{Coeq}_{\mathcal{A}lg_k}(f,g) \ .$$

Recall that when $\mathcal{S}$ is a symmetric monoidal category, then so is $\mathcal{M}on_{\mathcal{S}}$ (section 1.4).

Proposition 2.1.1. *Let $\mathcal{S}$ be a symmetric monoidal category. If $\mathcal{S}$ has equalizers then so does $\mathcal{M}on_{\mathcal{S}}$, and if $\mathcal{S}$ is regular then so is $\mathcal{M}on_{\mathcal{S}}$.*

Proof. Let $A \overset{f}{\underset{g}{\rightrightarrows}} A'$ be a parallel pair in $\mathit{Mon}_{\mathcal{S}}$. Let $E_0 \xrightarrow{\mathsf{can}} A$ be the equalizer of f and g in $\mathcal{S}$. Then there is a map $m_0 : E_0 {\otimes} E_0 \to E_0$ as below

$$\begin{tikzcd}
E_0{\otimes}E_0 \arrow[r, "\mathsf{can}{\otimes}\mathsf{can}"] \arrow[d, dashed, "m_0"'] & A{\otimes}A \arrow[r, shift left, "f{\otimes}f"] \arrow[r, shift right, "g{\otimes}g"'] \arrow[d, "m"] & A'{\otimes}A' \arrow[d, "m'"] \\
E_0 \arrow[r, "\mathsf{can}"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A'
\end{tikzcd}$$

because $fm(\mathsf{can}{\otimes}\mathsf{can}) = m'(f{\otimes}f)(\mathsf{can}{\otimes}\mathsf{can}) = m'(g{\otimes}g)(\mathsf{can}{\otimes}\mathsf{can}) = gm(\mathsf{can}{\otimes}\mathsf{can})$. Similarly there is a map $u_0 : I \to E_0$ as below

$$\begin{tikzcd}
E_0 \arrow[r, "\mathsf{can}"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A' \\
 & I \arrow[ul, dashed, "u_0"] \arrow[u, "u"] \arrow[ur, "u'"'] &
\end{tikzcd}$$

because $fu = u' = gu$. Then (E_0, m_0, u_0) becomes a monoid in $\mathcal{S}$; the associativity and unitality conditions follow from those for (A, m, u), plus the fact that can is a monomorphism (being an equalizer). For instance, let us check left unitality. In the following 3-d diagram, the front triangular face commutes by left unitality for A, the top rectangular face by definition of u_0, the lateral rectangular face by definition of m_0, and the hidden rectangular face by unitality in the monoidal category $\mathcal{S}$. Since can is monic, it follows that the bottom triangular face commutes, and this is left unitality for E_0.

$$\begin{tikzcd}
 & I{\otimes}E_0 \arrow[r, "u_0{\otimes}\mathsf{id}"] \arrow[dl, "\mathsf{id}{\otimes}\mathsf{can}"'] \arrow[dr, equal] & E_0{\otimes}E_0 \arrow[dl, "\mathsf{can}{\otimes}\mathsf{can}"'] \arrow[d, "m_0"] \\
I{\otimes}A \arrow[r, "u{\otimes}\mathsf{id}"] \arrow[dr, equal] & A{\otimes}A \arrow[d, "m"] & E_0 \arrow[dl, "\mathsf{can}"] \\
 & A &
\end{tikzcd}$$

Notice that by construction of m_0 and u_0, $\mathsf{can} : E_0 \to A$ is a morphism of monoids. Let $\alpha : M \to A$ be a morphism of monoids such that $f\alpha = g\alpha$. Then there is a unique morphism $\alpha_0 : M \to E_0$ such that $\mathsf{can}\alpha_0 = \alpha$. To conclude that $\mathsf{can} : E_0 \to A$ is the equalizer in $\mathit{Mon}_{\mathcal{S}}$ of f and g, it only remains to show that α_0 is a morphism of monoids. Again this follows from the corresponding fact for α, plus the fact that can is monic.

Finally, if $\mathcal{S}$ is regular, then so is $\mathit{Mon}_{\mathcal{S}}$, because if a morphism of monoids is invertible in $\mathcal{S}$, then it is clear that its inverse is also a morphism of monoids, hence the given map is an isomorphism in $\mathit{Mon}_{\mathcal{S}}$. ☐

Remarks 2.1.1.

1. We have just shown that the forgetful functor $\mathit{Mon}_{\mathcal{S}} \to \mathcal{S}$ creates isomorphims and equalizers. One can similarly show that it creates limits. This is [P, proposition 2.5].

2. Let $A \overset{f}{\underset{g}{\rightrightarrows}} B$ and $A' \overset{f'}{\underset{g'}{\rightrightarrows}} B'$ be two parallel pairs in a monoidal category $\mathcal{S}$. There is a canonical map

$$\mathsf{Eq}_{\mathcal{S}}(f,g) \otimes \mathsf{Eq}_{\mathcal{S}}(f',g') \xrightarrow{\mathsf{can} \otimes \mathsf{can}} \mathsf{Eq}_{\mathcal{S}}(f \otimes f', g \otimes g') \ .$$

   This map need not be an isomorphism, even when $\mathcal{S}$ is regular: consider the case $\mathcal{S} = \mathit{Vec}_k$, $f' = g' = g = 0$, $f = \mathsf{id}_A$. However, $\mathsf{can} \otimes \mathsf{can}$ is a monomorphism when $\mathcal{S}$ is regular, since it is the composite of $\mathsf{can} \otimes \mathsf{id}$ and $\mathsf{id} \otimes \mathsf{can}$, which are monic by regularity.

2.2 The 2-category of bicomodules

From now on $\mathcal{S}$ is assumed to be a regular monoidal category (section 2.1).

Recall (section 1.3) that k-algebras can be seen as the objects of a 2-category, whose arrows and 2-cells are bimodules and morphisms of bimodules respectively. Horizontal composition is tensor product of bimodules, which is defined as a certain coequalizer. This construction can be carried out more generally replacing Vec_k by any monoidal category where the tensor product preserves coequalizers. We are interested in the dual version of this: given a regular monoidal category $\mathcal{S}$, we will construct in this section a 2-category $\mathcal{G}$ as follows:

- the objects of $\mathcal{G}$ are the comonoids in $\mathcal{S}$
- the arrows $C \to D$ of $\mathcal{G}$ are the D-C-bicomodules in $\mathcal{S}$
- the 2-cells of $\mathcal{G}$ are the morphisms of bicomodules
- horizontal composition is the *tensor coproduct* of bicomodules.

The regularity assumption is needed in order to get an associative and unital tensor coproduct, as we shall see.

The tensor coproduct of bicomodules is a well-known notion in the context of k-coalgebra theory (that is when $\mathcal{S} = \mathit{Vec}_k$), see for instance [Mon, definition 8.4.2], where it is called "cotensor product" and denoted by the symbol $\square_C$. We will use the symbol $\otimes^C$ instead.

Definition 2.2.1. The tensor coproduct of a D-C-bicomodule (A_1, s_1, t_1) with a

C-E-bicomodule (A_2, s_2, t_2) is the following object of $\mathcal{S}$:

$$A_1 \otimes^C A_2 = \mathsf{Eq}_{\mathcal{S}}(A_1 \otimes A_2 \underset{\mathsf{id}_1 \otimes t_2}{\overset{s_1 \otimes \mathsf{id}_2}{\rightrightarrows}} A_1 \otimes C \otimes A_2) \quad .$$

First, we show that $A_1 \otimes^C A_2$ carries a natural structure of D-E-bicomodule. Notice that $(A_1 \otimes A_2, \mathsf{id}_1 \otimes s_2, t_1 \otimes \mathsf{id}_2)$ is a D-E-bicomodule; coassociativity and counitality follow from those for (A_1, t_1) and (A_2, s_2), and cocommutativity

$$\begin{tikzcd}
A_1 \otimes A_2 \arrow[r, "t_1 \otimes \mathsf{id}_2"] \arrow[d, "\mathsf{id}_1 \otimes s_2"'] & D \otimes A_1 \otimes A_2 \arrow[d, "\mathsf{id}_D \otimes \mathsf{id}_1 \otimes s_2"] \\
A_1 \otimes A_2 \otimes E \arrow[r, "t_1 \otimes \mathsf{id}_2 \otimes \mathsf{id}_E"'] & D \otimes A_1 \otimes A_2 \otimes E
\end{tikzcd}$$

is obvious.

Proposition 2.2.1. *There are unique maps* $s : A_1 \otimes^C A_2 \to (A_1 \otimes^C A_2) \otimes E$ *and* $t : A_1 \otimes^C A_2 \to D \otimes (A_1 \otimes^C A_2)$ *fitting in commutative diagrams:*

$$\begin{tikzcd}
A_1 \otimes^C A_2 \arrow[r, "can"] \arrow[d, dashed, "s"'] & A_1 \otimes A_2 \arrow[d, "id_1 \otimes s_2"] \\
(A_1 \otimes^C A_2) \otimes E \arrow[r, "can \otimes id_E"'] & A_1 \otimes A_2 \otimes E
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \otimes^C A_2 \arrow[r, "can"] \arrow[d, dashed, "t"'] & A_1 \otimes A_2 \arrow[d, "t_1 \otimes id_2"] \\
D \otimes (A_1 \otimes^C A_2) \arrow[r, "id_D \otimes can"'] & D \otimes A_1 \otimes A_2
\end{tikzcd} \quad .$$

In other words, $A_1 \otimes^C A_2$ *is a* D-E-*subbicomodule of* $A_1 \otimes A_2$ *via* can.

Proof. By regularity, $(A_1 \otimes^C A_2) \otimes E = \mathsf{Eq}(s_1 \otimes \mathsf{id}_2 \otimes \mathsf{id}_E, \mathsf{id}_1 \otimes t_2 \otimes \mathsf{id}_E)$. Therefore, to prove the existence and uniqueness of s, we need to show that

$$(s_1 \otimes \mathsf{id}_2 \otimes \mathsf{id}_E)(\mathsf{id}_1 \otimes s_2)\mathsf{can} = (\mathsf{id}_1 \otimes t_2 \otimes \mathsf{id}_E)(\mathsf{id}_1 \otimes s_2)\mathsf{can} \ .$$

Now, since A_2 is a bicomodule, s_2 and t_2 cocommute. This, and the definition of

$A_1 \otimes^C A_2$ as an equalizer, allows us to argue that

$$(\mathsf{id}_1 \otimes t_2 \otimes \mathsf{id}_E)(\mathsf{id}_1 \otimes s_2)\mathsf{can} = (\mathsf{id}_1 \otimes \mathsf{id}_C \otimes s_2)(\mathsf{id}_1 \otimes t_2)\mathsf{can}$$

$$= (\mathsf{id}_1 \otimes \mathsf{id}_C \otimes s_2)(s_1 \otimes \mathsf{id}_2)\mathsf{can} = (s_1 \otimes \mathsf{id}_2 \otimes \mathsf{id}_E)(\mathsf{id}_1 \otimes s_2)\mathsf{can}.$$

Similarly one shows the corresponding assertion for t. The bicomodule axioms for $A_1 \otimes^C A_2$ follow from those for $A_1 \otimes A_2$, plus the fact that can is monic. ☐

Now we define the tensor coproduct of morphisms. Let $f_1 : A_1 \to A_1'$ and $f_2 : A_2 \to A_2'$ be morphisms of D-C and C-E-bicomodules respectively. The two composites below

$$\begin{tikzcd} A_1 \otimes^C A_2 \arrow[r, "\mathsf{can}"] & A_1 \otimes A_2 \arrow[r, "f_1 \otimes f_2"] & A_1' \otimes A_2' \arrow[r, shift left, "s_1' \otimes \mathsf{id}_2'"] \arrow[r, shift right, "\mathsf{id}_1' \otimes t_2'"'] & A_1' \otimes C \otimes A_2' \end{tikzcd}$$

are respectively equal to, by assumption on f_1 and f_2,

$$\begin{tikzcd} A_1 \otimes^C A_2 \arrow[r, "\mathsf{can}"] & A_1 \otimes A_2 \arrow[r, shift left, "s_1 \otimes \mathsf{id}_2"] \arrow[r, shift right, "\mathsf{id}_1 \otimes t_2"'] & A_1 \otimes C \otimes A_2 \arrow[r, "f_1 \otimes \mathsf{id}_C \otimes f_2"] & A_1' \otimes C \otimes A_2' \end{tikzcd} ,$$

which are actually one same map by definition of $A_1 \otimes^C A_2$. It follows that there is a unique map $f_1 \otimes^C f_2$ fitting into the commutative diagram

$$\begin{tikzcd} A_1 \otimes^C A_2 \arrow[r, "\mathsf{can}"] \arrow[d, dashed, "f_1 \otimes^C f_2"'] & A_1 \otimes A_2 \arrow[d, "f_1 \otimes f_2"] \\ A_1' \otimes^C A_2' \arrow[r, "\mathsf{can}'"'] & A_1' \otimes A_2' \end{tikzcd} .$$

Functoriality of $\otimes^C$ follows from this uniqueness property plus the functoriality of $\otimes$. It is also clear that $f_1 \otimes^C f_2$ is a morphism of D-E-bicomodules. We have thus constructed a functor

$$\otimes^C : \mathsf{Bicomod}_{\mathbb{S}}(D, C) \times \mathsf{Bicomod}_{\mathbb{S}}(C, E) \to \mathsf{Bicomod}_{\mathbb{S}}(D, E) .$$

We next prove that the tensor coproduct is associative. To this end it is convenient to introduce the triple tensor coproduct of bicomodules as follows. Let C_i be a comonoid in $\mathcal{S}$ for $i = 0, 1, 2, 3$ and (A_i, s_i, t_i) be a C_{i-1}-C_i-bicomodule for $i = 1, 2, 3$. Then we define

$$A_1 \otimes^{C_1} A_2 \otimes^{C_2} A_3 = \mathsf{Eq}_{\mathcal{S}}\big(A_1 \otimes A_2 \otimes A_3 \underset{\mathsf{id}_1 \otimes t_2 \otimes t_3}{\overset{s_1 \otimes s_2 \otimes \mathsf{id}_3}{\rightrightarrows}} A_1 \otimes C_1 \otimes A_2 \otimes C_2 \otimes A_3\big) \ .$$

Proposition 2.2.2. *There are canonical isomorphisms of C_0-C_3-bicomodules*

$$(A_1 \otimes^{C_1} A_2) \otimes^{C_2} A_3 \cong A_1 \otimes^{C_1} A_2 \otimes^{C_2} A_3 \cong A_1 \otimes^{C_1} (A_2 \otimes^{C_2} A_3) \ .$$

Proof. We will show that the following composite

$$(A_1 \otimes^{C_1} A_2) \otimes^{C_2} A_3 \xrightarrow{\mathsf{can}} (A_1 \otimes^{C_1} A_2) \otimes A_3 \xrightarrow{\mathsf{can} \otimes \mathsf{id}_3} A_1 \otimes A_2 \otimes A_3$$

is the equalizer of $s_1 \otimes s_2 \otimes \mathsf{id}_3$ and $\mathsf{id}_1 \otimes t_2 \otimes t_3$. This will prove the first isomorphism claimed; the proof for the other is analogous.

To this end, consider the following diagram in $\mathcal{S}$

$$\begin{tikzcd}
(A_1 \otimes^{C_1} A_2) \otimes^{C_2} A_3 \arrow[r, "\mathsf{can}"] \arrow[d, "\mathsf{can} \otimes^{C_2} \mathsf{id}_3"'] & (A_1 \otimes^{C_1} A_2) \otimes A_3 \arrow[r, shift left, "s_{12} \otimes \mathsf{id}_3"] \arrow[r, shift right, "\mathsf{id}_{12} \otimes t_3"'] \arrow[d, "\mathsf{can} \otimes \mathsf{id}_3"'] & (A_1 \otimes^{C_1} A_2) \otimes C_2 \otimes A_3 \arrow[d, "\mathsf{can} \otimes \mathsf{id}_{C_2} \otimes \mathsf{id}_3"] \\
(A_1 \otimes A_2) \otimes^{C_2} A_3 \arrow[r, "\mathsf{can}"] \arrow[d, shift right, "\mathsf{id}_1 \otimes t_2 \otimes^{C_2} \mathsf{id}_3"'] \arrow[d, shift left, "s_1 \otimes \mathsf{id}_2 \otimes^{C_2} \mathsf{id}_3"] & A_1 \otimes A_2 \otimes A_3 \arrow[r, shift left, "\mathsf{id}_1 \otimes s_2 \otimes \mathsf{id}_3"] \arrow[r, shift right, "\mathsf{id}_1 \otimes \mathsf{id}_2 \otimes t_3"'] \arrow[d, shift right, "\mathsf{id}_1 \otimes t_2 \otimes \mathsf{id}_3"'] \arrow[d, shift left, "s_1 \otimes \mathsf{id}_2 \otimes \mathsf{id}_3"] & A_1 \otimes A_2 \otimes C_2 \otimes A_3 \arrow[d, shift right, "\mathsf{id}_1 \otimes t_2 \otimes \mathsf{id}_{C_2} \otimes \mathsf{id}_3"'] \arrow[d, shift left, "s_1 \otimes \mathsf{id}_2 \otimes \mathsf{id}_{C_2} \otimes \mathsf{id}_3"] \\
(A_1 \otimes C_1 \otimes A_2) \otimes^{C_2} A_3 \arrow[r, "\mathsf{can}"'] & A_1 \otimes C_1 \otimes A_2 \otimes A_3 \arrow[r, shift left, "\mathsf{id}_1 \otimes \mathsf{id}_{C_1} \otimes s_2 \otimes \mathsf{id}_3"] \arrow[r, shift right, "\mathsf{id}_1 \otimes \mathsf{id}_{C_1} \otimes \mathsf{id}_2 \otimes t_3"'] & A_1 \otimes C_1 \otimes A_2 \otimes C_2 \otimes A_3
\end{tikzcd}$$

Let us check the hypothesis in Johnstone's lemma 1.2.1. The diagrams commute as required, either by naturality of can, definition of s_{12} or functoriality of $\otimes$.

By regularity, the rows and columns of this diagram are equalizers, except perhaps for the first column. But this is automatic in view of lemma 1.2.2. Also, $(\mathsf{id}_1{\otimes}\mathsf{id}_{C_1}{\otimes}s_2{\otimes}\mathsf{id}_3, \mathsf{id}_1{\otimes}\mathsf{id}_{C_1}{\otimes}\mathsf{id}_2{\otimes}t_3)$ and $(\mathsf{id}_1{\otimes}t_2{\otimes}\mathsf{id}_3, s_1{\otimes}\mathsf{id}_2{\otimes}\mathsf{id}_{C_2}{\otimes}\mathsf{id}_3)$ are both coreflexive pairs, being split respectively by $\mathsf{id}_1{\otimes}\mathsf{id}_{C_1}{\otimes}\mathsf{id}_2{\otimes}\epsilon_{C_2}{\otimes}\mathsf{id}_3$ and $\mathsf{id}_1{\otimes}\epsilon_{C_1}{\otimes}\mathsf{id}_2{\otimes}\mathsf{id}_{C_2}{\otimes}\mathsf{id}_3$. Thus Johnstone's lemma applies, and it yields precisely the desired conclusion. ☐

Remark 2.2.1. One can use lemma 1.2.2 in the same way as in the proof above to deduce that the tensor coproduct over C preserves equalizers.

Notice that for any comonoid (C, Δ, ϵ), (C, Δ, Δ) becomes a C-C-bicomodule. Moreover, for any D-C-bicomodule (A, s, t), $s : A \to A{\otimes}C$ and $t : A \to D{\otimes}A$ are morphisms of D-C-bicomodules, precisely by definition of bicomodule. Now it is time for the unitality of the tensor coproduct.

Proposition 2.2.3. *Let (A, s, t) be a D-C-bicomodule. There are canonical isomorphisms of D-C-bicomodules as follows:*

$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[dr, dashed, "\cong"'] & A{\otimes}C & & A \arrow[r, "t"] \arrow[dr, dashed, "\cong"'] & D{\otimes}A \\
& A{\otimes}^{C}C \arrow[u, "\mathsf{can}"'] & & & D{\otimes}^{D}A \arrow[u, "\mathsf{can}"']
\end{tikzcd}$$

Proof. Let us check the assertion for s, that for t is analogous. By definition of ${\otimes}^{C}$, it is enough to show that $s : A \to A{\otimes}C$ is the equalizer of $s{\otimes}\mathsf{id}_C$ and $\mathsf{id}_A{\otimes}\Delta_C$. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "s"] & A{\otimes}C \arrow[r, shift left, "s{\otimes}\mathsf{id}_C"] \arrow[r, shift right, "\mathsf{id}_A{\otimes}\Delta_C"'] & A{\otimes}C{\otimes}C
\end{tikzcd}$$

Coassociativity for (A, s) says precisely that this diagram is a fork. Counitality says, moreover, that this fork is split by $\mathsf{id}_A{\otimes}\epsilon_C : A{\otimes}C \to A$ and $\mathsf{id}_A{\otimes}\mathsf{id}_C{\otimes}\epsilon_C : A{\otimes}C{\otimes}C \to A{\otimes}C$.

Hence this is an equalizer diagram by lemma 1.1.1. ☐

We can summarize the above results as follows.

Theorem 2.2.1. *Let $\mathcal{S}$ be a regular monoidal category. There is a 2-category $\mathcal{G}$ such that*

- *the objects of $\mathcal{G}$ are the comonoids in $\mathcal{S}$,*
- *the arrows $C \to D$ of $\mathcal{G}$ are the D-C-bicomodules in $\mathcal{S}$, the identity of C being the C-C-bicomodule $(C, \Delta_C, \epsilon_C)$,*
- *the 2-cells of $\mathcal{G}$ are the morphisms of bicomodules, with obvious vertical composition and identities, and*
- *horizontal composition is the tensor coproduct of bicomodules and their morphisms.*

Proof. The relevant work has already been done in the above propositions and constructions. For instance, the compatibility between the vertical and horizontal structures is precisely the functoriality of $\otimes^C$. ☐

2.3 Internal graphs and categories

Recall (section 1.4) that for any object C of a 2-category $\mathcal{G}$, there is a monoidal category $\mathcal{G}(C, C)$ whose objects and morphisms are respectively the arrows $C \to C$ of $\mathcal{G}$ and the 2-cells among them, and whose tensor product comes from horizontal composition in $\mathcal{G}$. From theorem 2.2.1 we thus deduce that:

Corollary 2.3.1. *Let C be a comonoid in a regular monoidal category $\mathcal{S}$. There is a monoidal category $\mathcal{G}_C$ consisting of C-C-bicomodules and their morphisms, with tensor product $\otimes^C$ and unit object (C, Δ_C, Δ_C).*

Proof. Take $\mathcal{G}_C = \mathcal{G}(C, C)$ in theorem 2.2.1. □

We thus arrive at the main definition of this work.

Definition 2.3.1. An object of the monoidal category $\mathcal{G}_C$ is called an *internal graph* or a *graph object* in $\mathcal{S}$. We refer to C as the *base* of the graph, or we say that the graph is *over* C. A monoid in $\mathcal{G}_C$ is called an *internal category* or a *category object* in $\mathcal{S}$.

We first discuss a few particular instances of these definitions.

Examples 2.3.1.

1. Let I be the unit object of $\mathcal{S}$. Then every object of $\mathcal{S}$ has a unique I-I-bicomodule structure, given by the identity of I. It follows that $\mathcal{G}_I = \mathcal{S}$, and thus a category over I is just a monoid in $\mathcal{S}$.

2. Let $\mathcal{S}$ be a lex category, that is a category with finite products and equalizers. As mentioned in section 1.4, any object X of $\mathcal{S}$ carries in this case a unique comonoid structure and, moreover, a graph over X is just an X-X-graded object. This is the usual definition of internal graph to a lex category, as for instance in [CPP]. In addition, it is immediate from the definition that the tensor coproduct of bicomodules (A_i, s_i, t_i) coincides with the pull-back of

bigraded objects:

$$A_1 \times^X A_2 = \{(a_1, a_2) \in A_1 \times A_2 \ / \ \tilde{s_1}(a_1) = \tilde{t_2}(a_2)\},$$

where $s_i(a) = (a, \tilde{s_i}(a))$ and $t_i(a) = (\tilde{t_i}(a), a)$, and $\tilde{s_i}$ and $\tilde{t_i}$ denote the degree maps (as explained in section 1.4). Therefore, our definition of internal category also reduces to the usual one [Joh 2.1, CPP] in this case. In particular, internal graphs and categories to ***Sets*** are respectively just small graphs and categories, in the usual sense of [ML, chapter I.2].

Before moving on to other basic examples, let us make explicit the conditions in definition 2.3.1. Recalling that the unit object of the category $\mathcal{G}_C$ of graphs over C is $(C, \Delta_C, \epsilon_C)$, we see that a category object $\mathfrak{C}$ in $\mathcal{S}$ is a 6-tuple $\mathfrak{C} = (A, C, s, t, i, m)$ where

- C is a comonoid in $\mathcal{S}$,
- (A, s, t) is a C-C-bicomodule, s being the right and t the left structures,
- $i : C \to A$ is a morphism of C-C-bicomodules,
- $m : A \otimes^C A \to A$ is a morphism of C-C-bicomodules,

and these are such that the following diagrams commute:

$$\begin{tikzcd}
C \otimes^C A \arrow[r, "i \otimes^C \mathsf{id}_A"] & A \otimes^C A \arrow[d, "m"] & A \otimes^C C \arrow[l, "\mathsf{id}_A \otimes^C i"'] \\
& A \arrow[ul, "\cong", "t"'] \arrow[ur, "\cong"', "s"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes^C A) \otimes^C A \cong A \otimes^C (A \otimes^C A) \arrow[r, "\mathsf{id}_A \otimes^C m"] \arrow[d, "m \otimes^C \mathsf{id}_A"'] & A \otimes^C A \arrow[d, "m"] \\
A \otimes^C A \arrow[r, "m"] & A
\end{tikzcd} .$$

We sometimes refer to (A, C, s, t, i, m) respectively as the "arrows, objects, source and target maps, identities and composition" of the internal category $\mathfrak{C}$. The requirements that i and m be morphisms of bicomodules can be described by saying that they preserve sources and targets. This is their precise meaning when $\mathcal{S} = \mathit{Sets}$ (or any lex category). The commutative diagrams above will be referred to as the unitality and associativity conditions for $\mathfrak{C}$. When explicit mention of part of the structure is not needed, we will abbreviate $\mathfrak{C} = (A, C, \dots)$.

2.4 Basic examples

Internal categories to monoidal categories encompass various different concepts such as linear categories [Mit], algebroids (called graphs in [Mal]), coalgebroids [Del] and bialgebroids [Rav, Mal], as displayed in table 2.1. From the point of view of this work, however, the most relevant applications to quantum group theory arise instead from internal categories to Vec_k (including linear categories), as we will see in chapter 9, or from internal categories to Alg_k, as we will see in chapter 10.

Let us explain the terminology and assertions in the table. We have already mentioned that a category in Sets is just an ordinary small category. If $\mathfrak{C} = (A, X, \dots)$ is a category in $\mathit{Monoids}$, the multiplication of A and X can be seen as a tensor product $\otimes$ on the arrows and objects of $\mathfrak{C}$; functoriality of $\otimes$ being equivalent to m being a morphism of monoids. This turns $\mathfrak{C}$ into a strict monoidal category, and one can proceed conversely.

Table 2.1: Instances of internal categories

$\mathbb{S}$	Comonoid C	Graph over C	Category over C
$\mathtt{Sets}$	set	usual small graph	usual small category
$\mathtt{Monoids}$	monoid		small strict monoidal category
$\mathtt{Groups}$	group		cat^1-group
$\mathtt{Vec}_k$	k-coalgebra C	C-C-bicomodule	see chapter 9
	when $C = kX$	linear graph over X	linear category over X
$\mathtt{Vec}_k^{op}$	k-algebra	C-C-bimodule	k-coalgebroid
$\mathtt{Alg}_k$	k-bialgebra	C-C-bicomodule-algebra	see chapter 10
$\mathtt{Alg}_k^{op}$	commutative k-algebra	k-algebroid	k-bialgebroid

Categories in $\mathtt{Groups}$ have been considered at various places in the literature; they can be described more succintly as *crossed modules* or as *cat*^{1}*-groups* [Lod1]. A generalization of this equivalence to other monoidal categories (replacing $\mathtt{Sets}$ by $\mathtt{Vec}_k$, for instance) is possible; this is the object of chapter 10.

Categories in $\mathtt{Vec}_k$ are studied in chapter 9, with an eye on applications to quantum groups. A k-linear category yields the simplest example of a category in $\mathtt{Vec}_k$: the base coalgebra is the *group-like* coalgebra kX, where X is the set of objects of the given category. See section 9.1 for more details.

Deligne has introduced the notion of a k-coalgebroid ([Del], also [Mal]). Comparing the definitions one sees immediately that this coincides with the notion of a category in $\mathtt{Vec}_k^{op}$.

Categories in $\mathtt{Alg}_k$ are studied in chapter 10, where the closely related notion of *cat*^{1}*-algebra* is defined, and the results on cat^1-groups described above extended.

Let us look at the case $\mathcal{S} = \mathtt{Alg}_k^{op}$ in detail. First, a comonoid in $\mathtt{Alg}_k^{op}$ is just a monoid in $\mathtt{Alg}_k$, and this is precisely a commutative k-algebra, by the well-known Hilton-Eckmann argument [ML, exercise II.5.5]. Now, let (A, K, s, t) be a graph in $\mathtt{Alg}_k^{op}$. Thus, K is a commutative k-algebra, A is a k-algebra, and $s : A \otimes K \to A$ and $t : K \otimes A \to A$ are morphisms of k-algebras that turn A into a K-K-bimodule. Define $\tilde{s} : K \to A$ by $\tilde{s}(x) = s(1 \otimes x)$ and $\tilde{t} : K \to A$ by $\tilde{t}(x) = t(x \otimes 1)$. Then $\tilde{s} = s \circ (\mathsf{u}_A \otimes \mathsf{id}_K)$ is a morphism of algebras, and by unitality $s(a \otimes 1) = a$, hence

$$\tilde{s}(x)a = s(1 \otimes x)s(a \otimes 1) = s(a \otimes x) = s(a \otimes 1)s(1 \otimes x) = a\tilde{s}(x),$$

i.e. $\mathsf{Im}\tilde{s} \subseteq Z(A)$. Similarly, $\tilde{t} : K \to A$ is a morphism of algebras and $\mathsf{Im}\tilde{t} \subseteq$

$Z(A)$. Therefore, $(A, K, \tilde{s}, \tilde{t})$ is precisely what Ravenel calls a K-algebroid [Rav] (and Maltsiniotis a "graph" [Mal]). Conversely, any K-algebroid comes from a unique graph in Alg_k^{op} as above, so the concepts are equivalent. Finally, since a K-bialgebroid [Rav, Mal] can be defined as a comonoid in the category of K-algebroids, it follows that K-bialgebroids and categories in Alg_k^{op} are equivalent concepts too.

We next describe the simplest examples of internal categories.

Examples 2.4.1.

1. For any comonoid C in $\mathcal{S}$, there is a category in $\mathcal{S}$

   $$\widehat{\mathrm{C}} = (C, C, \Delta_C, \Delta_C, \mathsf{id}_C, \Delta_C^{-1})$$

   called the *discrete* category on C. (Recall from proposition 2.2.3 that $\Delta_C : C \to C \otimes^C C$ is an isomorphism). When $\mathcal{S} = \mathit{Sets}$, the only arrows of this category are the identities.

2. For any monoid A in $\mathcal{S}$ there is a *one-object* category in $\mathcal{S}$

   $$\underset{\vee}{\mathrm{A}} = (A, I, \mathsf{id}_A, \mathsf{id}_A, \mathsf{u}_A, \mu_A) \; .$$

   Together with the previous one, this example shows that category objects generalize at the same time the notions of monoids and comonoids.

3. The *one-arrow* category in $\mathcal{S}$ is $\mathfrak{I} = \hat{\mathrm{I}} = \underset{\vee}{\mathrm{I}}$ where I is the unit object of I.

4. For any comonoid C in $\mathcal{S}$, the *pair* or *coarse* category on C in $\mathcal{S}$ is

   $$\underset{\vee}{\widehat{\mathrm{C}}} = (C \otimes C, C, \mathsf{id}_C \otimes \Delta_C, \Delta_C \otimes \mathsf{id}_C, \Delta_C, \mathsf{id}_C \otimes \epsilon_C \otimes \epsilon_C \otimes \mathsf{id}_C) \; .$$

5. The definition of internal categories is as flexible as to admit as an example the following category with "no arrows at all". Suppose $\mathcal{S}$ has a zero object 0 such that $0 \otimes V = 0 = V \otimes 0$ for every object V of $\mathcal{S}$. This is the case for instance when $\mathcal{S} = \mathbf{Vec}_k$. Then for every comonoid C in $\mathcal{S}$ there is defined the *empty category* over C as

$$\Phi_C = (0, C, s, t, i, m)$$

where all s, t, i and m are zero. The axioms hold trivially. Notice that in this case i is not a monomorphism, unlike the case of lex categories (where i is split by both $\tilde{s}$ and $\tilde{t}$).

Chapter 3

Corestriction and coinduction of comodules

The familiar adjunction between restriction and induction for modules over k-algebras holds in fact for arbitrary monoidal categories. We prove below the dual version of this result, since it is mostly in this form that we will use it later.

Let $f : C \to D$ be a morphism of comonoids in a monoidal category $\mathcal{S}$.

If (M, t) is a left C-comodule, we let ${}_fM$ denote the same object M but viewed as left D-comodule via the map

$$M \xrightarrow{t} C \otimes M \xrightarrow{f \otimes \mathsf{id}_M} D \otimes M \ .$$

We say that ${}_fM$ is obtained from M by *corestriction via f*.

Corestriction is a functor

$$\mathsf{cores}_f : \mathsf{Comod}_{\mathcal{S}} C \to \mathsf{Comod}_{\mathcal{S}} D, \quad M \mapsto {}_fM \ .$$

Right corestriction is defined similarly; if M is a right C-comodule, we let M_f denote the corresponding right D-comodule. More generally, if $f' : C' \to D'$ is another morphism of comonoids, there is the two-sided corestriction functor

$$\mathsf{cores}_{f,f'} : \mathsf{Bicomod}_{\mathcal{S}}(C, C') \to \mathsf{Bicomod}_{\mathcal{S}}(D, D'), \quad M \mapsto {}_fM_{f'} \; .$$

Assume now that $\mathcal{S}$ is a regular monoidal category, so that the tensor coproduct of bicomodules is well-defined and associative (section 2.2).

Lemma 3.0.1. *Let X be a right C-comodule and Y a left one. Then there is a natural monomorphism $X \otimes^C Y \to X_f \otimes^D {}_fY$ making*

$$\begin{tikzcd}
X \otimes^C Y \arrow[rr, dashed] \arrow[dr, "\mathsf{can}"'] & & X_f \otimes^D {}_fY \arrow[dl, "\mathsf{can}"] \\
& X \otimes Y &
\end{tikzcd}$$

commutative.

Proof. The dotted arrow can be filled in because the squares below commute by definition of corestriction.

$$\begin{tikzcd}
X \otimes^C Y \arrow[r, "\mathsf{can}"] \arrow[d, dashed] & X \otimes Y \arrow[r, shift left, "s \otimes \mathsf{id}_Y"] \arrow[r, shift right, "\mathsf{id}_X \otimes t"'] \arrow[d, equal] & X \otimes C \otimes Y \arrow[d, "\mathsf{id}_X \otimes f \otimes \mathsf{id}_Y"] \\
X_f \otimes^D {}_fY \arrow[r, "\mathsf{can}"'] & X_f \otimes {}_fY \arrow[r, shift left, "s_{X_f} \otimes \mathsf{id}_Y"] \arrow[r, shift right, "\mathsf{id}_X \otimes t_{{}_fY}"'] & X \otimes D \otimes Y
\end{tikzcd}$$

□

The C-D-bicomodule C_f is used to define the *coinduction* functor as follows:

$$\mathsf{coind}_f : \mathsf{Comod}_{\mathcal{S}} D \to \mathsf{Comod}_{\mathcal{S}} C, \quad N \mapsto C_f \otimes^D N \; .$$

Proposition 3.0.1. *coind_f is right adjoint to cores_f.*

Proof. We have to show that

$$\mathsf{Hom}_C(M, C_f\otimes^D N) \cong \mathsf{Hom}_D({}_fM, N)$$

naturally in $M \in \mathsf{Comod}_S C$ and $N \in \mathsf{Comod}_S D$.

The correspondences are as follows: given $u \in \mathsf{Hom}_C(M, C_f\otimes^D N)$, one defines $\tilde{u} \in \mathsf{Hom}_D({}_fM, N)$ as the composite

$$\tilde{u} : {}_fM \xrightarrow{u} C_f\otimes^D N \xrightarrow{\epsilon_C\otimes\mathsf{id}_N} N \ ,$$

and given $v \in \mathsf{Hom}_D({}_fM, N)$ one defines $\hat{v} \in \mathsf{Hom}_C(M, C_f\otimes^D N)$ as

$$\hat{v} : M \xrightarrow{t} C\otimes^C M \to C_f\otimes^D {}_fM \xrightarrow{\mathsf{id}_C\otimes^D v} C_f\otimes^D N \ ,$$

where we have made use of the canonical maps of lemmas 2.2.3 and 3.0.1.

Let us check that ˜ and ˆ are in fact inverse correspondences, but omit the proof that $\tilde{u}$ and $\hat{v}$ are morphisms of comodules as claimed, which is similar.

First, $\tilde{\hat{v}} = v$ simply by counitality for (M, t). On the other hand, the commutativity of the following diagram shows that $\hat{\tilde{u}} = u$:

$$\begin{tikzcd}
M \arrow[r, "t"] \arrow[dr, "u"'] & C\otimes^C M \arrow[r] \arrow[dr, "\mathsf{id}_C\otimes^C u"] & C_f\otimes^D {}_fM \arrow[r, "\mathsf{id}_C\otimes^D u"] & C_f\otimes^D {}_fC_f\otimes^D N \arrow[r, "\mathsf{id}_C\otimes\epsilon_C\otimes\mathsf{id}_N"] & C_f\otimes N \\
 & C_f\otimes^D N \arrow[r, "\Delta_C\otimes^D\mathsf{id}_N"] \arrow[urrr, bend right, "\mathsf{id}"'] & C\otimes^C C_f\otimes^D N \arrow[ur] & &
\end{tikzcd}$$

$\square$

Remarks 3.0.1.

1. Suppose that M and N are C-E-and D-E-bicomodules respectively. Then the proof above also shows that there is a natural bijection

$$\mathsf{Hom}_{C\text{–}E}(M, C_f\otimes^D N) \cong \mathsf{Hom}_{D\text{–}E}({}_fM, N) \ .$$

2. The right version of this result holds as well:

$$\mathsf{Hom}_{E\text{–}C}(M, N \otimes^{D}{}_{f} C) \cong \mathsf{Hom}_{E\text{–}D}(M_f, N) \ ,$$

naturally in $M \in \mathsf{Bicomod}_{\mathcal{S}}(E, C)$ and $N \in \mathsf{Bicomod}_{\mathcal{S}}(E, D)$. A particular instance of this that will get used many times later is:

$$\mathsf{Hom}_C(C, N \otimes C) \cong \mathsf{Hom}_{\mathcal{S}}(C, N) \text{ under } u \mapsto (\mathsf{id}_N \otimes \epsilon_C) \circ u \ ,$$

where N is any object of $\mathcal{S}$ and Hom_C denotes morphisms of right C-comodules. (To deduce it from the previous one, take $D = I$, $f = \epsilon_C$).

Chapter 4

Functors and cofunctors

4.1 Functors and natural transformations

In this section we introduce the most natural (but, from the point of view of this work, not the most useful) notion of morphism between internal categories; namely, functors, along with their natural transformations.

Definition 4.1.1. Let $\mathfrak{C} = (A, C, \dots)$ and $\mathfrak{D} = (B, D, \dots)$ be categories in $\mathcal{S}$. A functor $f : \mathfrak{C} \to \mathfrak{D}$ is a pair $f = (f_1, f_0)$ where

- $f_0 : C \to D$ is a morphism of comonoids in $\mathcal{S}$,
- $f_1 : A \to B$ is a morphism in $\mathcal{S}$,

and the following diagrams commute

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "s"'] & B \arrow[d, "s"] \\
A \otimes C \arrow[r, "f_1 \otimes f_0"'] & B \otimes D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "t"'] & B \arrow[d, "t"] \\
C \otimes A \arrow[r, "f_0 \otimes f_1"'] & D \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f_1"] & B \\
C \arrow[u, "i"] \arrow[r, "f_0"'] & D \arrow[u, "i"']
\end{tikzcd}$$

$$\begin{tikzcd} A{\otimes}^{C}A \arrow[r, hook] \arrow[d, "m"'] & A{\otimes}^{D}A \arrow[r, "f_1{\otimes}^{D}f_1"] & B{\otimes}^{D}B \arrow[d, "m"] \\ A \arrow[rr, "f_1"'] & & B \end{tikzcd} .$$

Above, A is viewed as D-bicomodule by corestriction via f_0. Hence the canonical map $A{\otimes}^{C}A \to A{\otimes}A$ maps to $A{\otimes}^{D}A$. The first two diagrams say that $f_1 : A \to B$ is a morphism of D-bicomodules, hence $f_1{\otimes}^{D}f_1 : A{\otimes}^{D}A \to B{\otimes}^{D}B$ is defined.

The composition of two functors $f : \mathfrak{C} \to \mathfrak{D}$ and $g : \mathfrak{D} \to \mathfrak{E}$ is the functor $h = g \circ f : \mathfrak{C} \to \mathfrak{E}$ defined by $h_0 = g_0 \circ f_0$ and $h_1 = g_1 \circ f_1$. The identity functor of $\mathfrak{C}$ is $\mathsf{id}_{\mathfrak{C}} = (\mathsf{id}_A, \mathsf{id}_C)$. We use $\overrightarrow{\mathit{Cat}}_{\mathcal{S}}$ to denote the category whose objects are the internal categories to $\mathcal{S}$ with functors as morphisms.

A category $\mathfrak{C}$ is said to be *augmented* if it admits a functor $\epsilon : \mathfrak{C} \to \mathfrak{I}$. In this case necessary $\epsilon_0 = \epsilon_C$, since this is the unique morphism of comonoids $C \to I$. If $\mathcal{S}$ is lex then every category is uniquely augmented. In general a category need not be augmented, see below.

Examples 4.1.1.

1. Let C and D be two comonoids in $\mathcal{S}$. If $f : C \to D$ is a morphism of comonoids then $(f, f) : \widehat{\mathrm{C}} \to \widehat{\mathrm{D}}$ is a functor; conversely, any functor $\widehat{\mathrm{C}} \to \widehat{\mathrm{D}}$ is of that form for some f. This yields a fully-faithful functor $\mathit{Comon}_{\mathcal{S}} \to \overrightarrow{\mathit{Cat}}_{\mathcal{S}}$. In particular there is a unique functor $\epsilon : \widehat{\mathrm{C}} \to \mathfrak{I}$, given by $\epsilon = (\epsilon_C, \epsilon_C)$. Thus $\widehat{\mathrm{C}}$ is augmented.

2. Let A and B be two monoids in $\mathcal{S}$. If $f : A \to B$ is a morphism of monoids then $(f, \mathsf{id}_I) : \underset{\vee}{\mathrm{A}} \to \underset{\vee}{\mathrm{B}}$ is a functor; conversely, any functor $\underset{\vee}{\mathrm{A}} \to \underset{\vee}{\mathrm{B}}$ is of that form

for some f. This yields a fully-faithful functor $\mathcal{Mon}_{\mathbb{S}} \to \overrightarrow{\mathcal{Cat}}_{\mathbb{S}}$. In particular there is a unique functor $u : \mathfrak{I} \to \underset{\vee}{\mathrm{A}}$, given by $u = (\mathsf{u}_A, \mathsf{id}_I)$. Not every monoid A admits a morphism to I. Hence not every category is augmented.

3. $\underset{\vee}{\widehat{\mathrm{C}}}$ is augmented via $(\epsilon_C \otimes \epsilon_C, \epsilon_C)$.

4. For any category $\mathfrak{C}$ over C in $\mathbb{S}$ there is a functor $\widehat{\mathrm{C}} \to \mathfrak{C}$ given by (i, id_C). If $\mathfrak{C}$ is augmented, with augmentation ϵ, then there is also a functor $\mathfrak{C} \to \underset{\vee}{\widehat{\mathrm{C}}}$ given by $\mathsf{id}_C : C \to C$ and $A \xrightarrow{s} A \otimes C \xrightarrow{t \otimes \mathsf{id}_C} C \otimes A \otimes C \xrightarrow{\mathsf{id}_C \otimes \epsilon_1 \otimes \mathsf{id}_C} C \otimes C$. Conversely, if there is a functor $f : \mathfrak{C} \to \underset{\vee}{\widehat{\mathrm{C}}}$ then $\mathfrak{C}$ is augmented via $\epsilon_{\underset{\vee}{\widehat{\mathrm{C}}}} \circ f$.

Definition 4.1.2. Let $f, g : \mathfrak{C} \to \mathfrak{D}$ be two functors. A natural transformation $\alpha : f \Rightarrow g$ is a morphism $\alpha : C \to B$ in $\mathbb{S}$ such that the following diagrams commute

$$\begin{tikzcd}
C \arrow[r, "\alpha"] \arrow[d, "\Delta_C"'] & B \arrow[d, "s"] \\
C \otimes C \arrow[r, "\alpha \otimes f_0"'] & B \otimes D
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\alpha"] \arrow[d, "\Delta_C"'] & B \arrow[d, "t"] \\
C \otimes C \arrow[r, "g_0 \otimes \alpha"'] & D \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
 & A \otimes^C C \arrow[r, hook] & A_g \otimes^D {}_g C \arrow[r, "g_1 \otimes^D \alpha"] & B \otimes^D B \arrow[dr, "m"] & \\
A \arrow[ur, "s", "\cong"'] \arrow[dr, "t"', "\cong"] & & & & B \\
 & C \otimes^C A \arrow[r, hook] & C_f \otimes^D {}_f A \arrow[r, "\alpha \otimes^D f_1"'] & B \otimes^D B \arrow[ur, "m"'] &
\end{tikzcd}$$

To explain the notation in the last diagram, notice that the first two diagrams say that $\alpha : {}_g C_f \to B$ is a morphism of D-D-bicomodules, where the subindices denote corestriction via (g_0, f_0) as in chapter 3. Recall also that, by definition of functor, $f_1 : {}_fA_f :\longrightarrow B$ is a morphism of D-D-bicomodules, and similarly for g; hence, the maps $g_1 \otimes^D \alpha$ and $\alpha \otimes^D f_1$ in the third diagram are well-defined. We have also made use of the canonical map of lemma 3.0.1.

The identity natural transformation $\mathsf{id}_f : f \Rightarrow f$ is defined by the map $C \xrightarrow{f_0} D \xrightarrow{i} B$. In this case the commutativity of diagrams 4.1.2 follows from the definition of functor for f and the unitality property for $\mathfrak{C}$.

Now consider three functors $f, g, h : \mathfrak{C} \to \mathfrak{D}$ and natural transformations $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$. The composition $\beta \circ \alpha : f \Rightarrow h$ is defined by the composite map

$$C \xrightarrow{\Delta_C} C \otimes^C C \hookrightarrow C_g \otimes^D {}_gC \xrightarrow{\beta \otimes^D \alpha} B \otimes^D B \xrightarrow{m} B \ ;$$

where we have made use of the canonical map of lemma 3.0.1. Let us check the axioms for a natural transformation in detail. As explained above, the commutativity of the first two diagrams in 4.1.2 is equivalent to the fact that $\beta \circ \alpha : {}_hC_f \to B$ be a morphism of D-D-bicomodules. To see that this is the case we make explicit the bicomodule structures involved in the definition of $\beta \circ \alpha$ as a composition:

$${}_hC_f \xrightarrow{\Delta_C} {}_hC \otimes^C C_f \hookrightarrow {}_hC_g \otimes^D {}_gC_f \xrightarrow{\beta \otimes^D \alpha} B \otimes^D B \xrightarrow{m} B \ .$$

Since m, α, β and Δ_C are all morphisms of D-D-bicomodules as indicated, so is $\beta \circ \alpha$. The remaining condition is the commutativity of the diagram below, which holds by

(1) coassociativity of s and t

(2) s and t cocommute

(3) α and β are natural transformations

(4) associativity of m.

$$\begin{tikzcd}
A\otimes^D C \arrow[rr, "\mathsf{id}_A\otimes^D\Delta_C"] & & A\otimes^D C\otimes^D C \arrow[rr, "h_1\otimes^D\beta\otimes^D\alpha"] & & B\otimes^D B\otimes^D B \arrow[rr, "\mathsf{id}_B\otimes^D m"] \arrow[d, "m\otimes^D\mathsf{id}_B"] & & B\otimes^D B \arrow[dd, "m"] \\
 & (1) & A\otimes^D C \arrow[u, swap, "s\otimes^D\mathsf{id}_C"] \arrow[d, "t\otimes^D\mathsf{id}_C"] & (3) & B\otimes^D B \arrow[drr, "m"] & (4) & \\
A \arrow[uu, "s"] \arrow[urr, "s"] \arrow[drr, swap, "t"] \arrow[dd, swap, "t"] & (2) & C\otimes^D A\otimes^D C \arrow[rr, "\beta\otimes^D g_1\otimes^D\alpha"] & & B\otimes^D B\otimes^D B \arrow[u, "m\otimes^D\mathsf{id}_B"] \arrow[d, swap, "\mathsf{id}_B\otimes^D m"] & (4) & B \\
 & (1) & C\otimes^D A \arrow[u, swap, "\mathsf{id}_C\otimes^D s"] \arrow[d, swap, "\mathsf{id}_C\otimes^D t"] & (3) & B\otimes^D B \arrow[urr, swap, "m"] & (4) & \\
C\otimes^D A \arrow[rr, swap, "\Delta_C\otimes^D\mathsf{id}_A"] & & C\otimes^D C\otimes^D A \arrow[rr, swap, "\beta\otimes^D\alpha\otimes^D f_1"] & & B\otimes^D B\otimes^D B \arrow[u, swap, "\mathsf{id}_B\otimes^D m"] \arrow[rr, swap, "m\otimes^D\mathsf{id}_B"] & & B\otimes^D B \arrow[uu, swap, "m"]
\end{tikzcd}$$

One can similarly define the *horizontal* composition of a natural transformation with a functor. One obtains in this way a 2-category of internal categories, functors and natural transformations, that we still denote by $\overrightarrow{\mathcal{C}at}_{\mathbb{S}}$.

We next describe the monoid of endomorphisms in $\overrightarrow{\mathcal{C}at}_{\mathbb{S}}$ of the identity functor $\mathsf{id}_{\mathfrak{C}}$ for some simple choices of $\mathfrak{C}$. It is easy to see that $\mathsf{End}(\mathsf{id}_{\mathfrak{C}})$ is always a commutative monoid; this is a general fact for 2-categories. Later we will prove the stronger assertion that $\mathsf{End}(\mathsf{id}_{\mathfrak{C}})$ is in the center of the monoid of admissible sections of $\mathfrak{C}$ (corollary 5.3.1).

Examples 4.1.2.

For the functor $\epsilon : \widehat{\mathrm{C}} \to \mathfrak{I}$ we have $\mathsf{End}(\epsilon) = \mathsf{Hom}_{\mathbb{S}}(C, I)$; the naturality of any such map $C \to I$ boils down to the counitality of C. In this case the composition of natural transformations coincides with the convolution product in $\mathsf{Hom}_{\mathcal{S}}(C, I)$. By generalities on 2-categories, ϵ induces a morphism of monoids

$$\mathsf{End}(\mathsf{id}_{\widehat{C}}) \to \mathsf{End}(\epsilon \circ \mathsf{id}_{\widehat{C}}) = \mathsf{Hom}_{\mathcal{S}}(C, I),\ \alpha \mapsto \epsilon_C \circ \alpha\ .$$

It is easy to see that this is an isomorphism onto $\{\varphi \in \mathsf{Hom}_{\mathcal{S}}(C, I)\ /\ (\varphi \otimes \mathsf{id}_C) \circ \Delta_C = (\mathsf{id}_C \otimes \varphi) \circ \Delta_C\}$, which is a submonoid of the center of $\mathsf{Hom}_{\mathcal{S}}(C, I)$.

For the functor $u : \mathfrak{I} \to \underset{\vee}{A}$ we have $\mathsf{End}(u) = \mathsf{Hom}_{\mathcal{S}}(I, A)$; the naturality of any such map $I \to A$ boils down to the unitality of A. In this case the composition of natural transformations coincides with the convolution product in $\mathsf{Hom}_{\mathcal{S}}(I, A)$. By generalities on 2-categories, u induces a morphism of monoids

$$\mathsf{End}(\mathsf{id}_{\underset{\vee}{A}}) \to \mathsf{End}(\mathsf{id}_{\underset{\vee}{A}} \circ u) = \mathsf{Hom}_{\mathcal{S}}(I, A),\ \alpha \mapsto \alpha \circ \mathsf{id}_I\ ,$$

which turns out to be the inclusion. It is easy to see that in fact $\mathsf{End}(\mathsf{id}_{\underset{\vee}{A}}) = \{a \in \mathsf{Hom}_{\mathcal{S}}(I, A)\ /\ \mu_A \circ (a \otimes \mathsf{id}_A) = \mu_A \circ (\mathsf{id}_A \otimes a)\}$, which is a submonoid of the center of $\mathsf{Hom}_{\mathcal{S}}(I, A)$.

4.2 Cofunctors and natural cotransformations

In this section we introduce an alternative notion of morphism for internal categories, with respect to which most later constructions will be functorial. This is one of the most important technical notions in this work.

Definition 4.2.1. Let $\mathfrak{C} = (A, C, \dots)$ and $\mathfrak{D} = (B, D, \dots)$ be categories in $\mathcal{S}$. A cofunctor $\varphi : \mathfrak{C} \to \mathfrak{D}$ is a pair $\varphi = (\varphi_1, \varphi_0)$ where

- $\varphi_0 : D \to C$ is a morphism of comonoids in $\mathcal{S}$,
- $\varphi_1 : A \otimes^C D \to B$ is a morphism of C-D-bicomodules in $\mathcal{S}$ (where B and D are viewed as left C-comodules by corestriction via φ_0),

and the following diagrams commute

$$\begin{tikzcd}
A\otimes^C D \arrow[r, "\varphi_1"] & B \\
C\otimes^C D \arrow[u, "i\otimes^C \mathsf{id}_D"] & D \arrow[l, "\cong"'] \arrow[u, "i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes^C A\otimes^C D \arrow[r, "\mathsf{id}_A\otimes^C \varphi_1"] \arrow[d, "m\otimes^C \mathsf{id}_D"'] & A\otimes^C B \cong A\otimes^C D\otimes^D B \arrow[r, "\varphi_1\otimes^D \mathsf{id}_B"] & B\otimes^D B \arrow[d, "m"] \\
A\otimes^C D \arrow[rr, "\varphi_1"'] & & B
\end{tikzcd} .$$

Cofunctors can be composed. Given two cofunctors $\varphi : \mathfrak{C} \to \mathfrak{D}$ and $\psi : \mathfrak{D} \to \mathfrak{E}$, where $\mathfrak{C} = (P, C, \dots)$, $\mathfrak{D} = (Q, D, \dots)$ and $\mathfrak{E} = (R, E, \dots)$, the composite $\rho = \psi \circ \varphi : \mathfrak{C} \to \mathfrak{E}$ is defined as $\rho = (\rho_1, \rho_0)$ where

$$\rho_0 : E \xrightarrow{\psi_0} D \xrightarrow{\varphi_0} C \text{ and}$$

$$\rho_1 : P\otimes^C E \cong P\otimes^C D\otimes^D E \xrightarrow{\varphi_1\otimes^D \mathsf{id}_E} Q\otimes^D E \xrightarrow{\psi_1} R .$$

Composition of cofunctors is associative and has identities: the identity cofunctor of $\mathfrak{C}$ is $(\mathsf{id}_C, \mathsf{id}_P)$. We use $\overleftarrow{\mathcal{C}at}_{\mathcal{S}}$ to denote the category whose objects are the internal categories to $\mathcal{S}$ with cofunctors as morphisms.

Remark 4.2.1. Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories over the same comonoid $C = D$. A cofunctor $\varphi = (\varphi_1, \varphi_0) : \mathfrak{C} \to \mathfrak{D}$ with $\varphi_0 = \mathsf{id}_C$ is the same as a functor $f = (f_1, f_0) : \mathfrak{C} \to \mathfrak{D}$ with $f_0 = \mathsf{id}_C$, via $\begin{tikzcd} A\otimes^C C \arrow[r, "\varphi_1"] & B \\ A \arrow[u, "t", "\cong"'] \arrow[ur, "f_1"'] & \end{tikzcd}$; the cofunctor and functor axioms correspond to each other in this case. Similarly, a cofunctor $\varphi = (\varphi_1, \varphi_0) : \mathfrak{C} \to \mathfrak{D}$

with φ_0 invertible is the same as a functor $f = (f_1, f_0) : \mathfrak{C} \to \mathfrak{D}$ with $f_0 = \varphi_0^{-1}$. In particular, isomorphisms in $\overleftarrow{\mathcal{C}at}_{\mathcal{S}}$ coincide with isomorphisms in $\overrightarrow{\mathcal{C}at}_{\mathcal{S}}$.

Examples 4.2.1.

1. Let C and D be two comonoids in $\mathcal{S}$. Let $\varphi_0 : D \to C$ be a morphism of comonoids. Then there is a cofunctor $(\varphi_1, \varphi_0) : \widehat{\mathrm{C}} \to \widehat{\mathrm{D}}$ where φ_1 is the canonical isomorphism $C \otimes^C D \to D$. Conversely, any cofunctor $\widehat{\mathrm{C}} \to \widehat{\mathrm{D}}$ is of this form for some φ_0. This yields a fully-faithful functor $(\mathcal{C}omon_{\mathcal{S}})^{op} \to \overleftarrow{\mathcal{C}at}_{\mathcal{S}}$. In particular there is a unique cofunctor $\mathfrak{I} \to \widehat{\mathrm{C}}$.

2. There is a fully-faithful functor $\mathcal{M}on_{\mathcal{S}} \to \overleftarrow{\mathcal{C}at}_{\mathcal{S}}$. This follows from the corresponding fact for $\overrightarrow{\mathcal{C}at}_{\mathcal{S}}$ (examples 4.1.1), plus remark 4.2.1 that functors over the identity are the same as cofunctors over the identity. In particular there is a unique cofunctor $\mathfrak{I} \to \underset{\vee}{\mathrm{A}}$.

3. For any category $\mathfrak{C}$, there is a unique cofunctor $\overleftarrow{i} : \mathfrak{I} \to \mathfrak{C}$ given by the pair $C \xrightarrow{\epsilon_C} I$ and $I \otimes^I C \cong C \xrightarrow{i} A$.

Definition 4.2.2. Let $\varphi, \psi : \mathfrak{C} \to \mathfrak{D}$ be two cofunctors. A natural cotransformation $\alpha : \varphi \Rightarrow \psi$ is a morphism $\alpha : D \to B$ in $\mathcal{S}$ such that the following diagrams commute

$$\begin{tikzcd}
D \arrow[r, "\alpha"] \arrow[d, "\Delta_D"'] & B \arrow[d, "s"] \\
D \otimes D \arrow[r, "\alpha \otimes \mathsf{id}_D"'] & B \otimes D
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[r, "\alpha"] \arrow[d, "\Delta_D"'] & B \arrow[d, "t"] \\
D \otimes D \arrow[d, "\varphi_0 \otimes \mathsf{id}_D"'] & D \otimes B \arrow[d, "\psi_0 \otimes \mathsf{id}_B"] \\
C \otimes D \arrow[r, "\mathsf{id}_C \otimes \alpha"'] & C \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
 & A\otimes^{C}{}_{\psi}B \arrow[r, "\cong"] & A\otimes^{C}{}_{\psi}D\otimes^{D}B \arrow[r, "\psi_1\otimes^{D}\mathsf{id}_B"] & B\otimes^{D}B \arrow[dr, "m"] & \\
A\otimes^{C}{}_{\varphi}D \arrow[ur, "\mathsf{id}_A\otimes^{C}\alpha"] \arrow[dr, "\varphi_1"'] & & & & B \\
 & B \arrow[r, "\cong"'] & D\otimes^{D}B \arrow[r, "\alpha\otimes^{D}\mathsf{id}_B"'] & B\otimes^{D}B \arrow[ur, "m"'] &
\end{tikzcd}$$

To explain the subindices in the last diagram, notice that the first diagram says that $\alpha : D \to B$ is a morphism of right D-comodules, while the second one that $\alpha : {}_{\varphi}D \to {}_{\psi}B$ is a morphism of left C-comodules; these subindices denote left corestriction via φ_0 and ψ_0 respectively, as in chapter 3. Recall also that, by definition of cofunctor, $\psi_1 : A\otimes^{C}{}_{\psi}D \to B$ is a morphism of right D-comodules, and similarly for φ; hence, the maps in the third diagram are well-defined.

The identity natural cotransformation $\mathsf{id}_{\varphi} : \varphi \Rightarrow \varphi$ is defined by the map $i : C \to A$. The commutativity of diagrams 4.2.2 is easy to check in this case.

Now consider three cofunctors $\varphi,\ \psi\ , \rho : \mathfrak{C} \to \mathfrak{D}$ and natural cotransformations $\alpha : \varphi \Rightarrow \psi$ and $\beta : \psi \Rightarrow \rho$. The composition $\beta \circ \alpha : \varphi \Rightarrow \rho$ is defined by the composite map

$$D \xrightarrow{\alpha} B \xrightarrow{t} D\otimes^{D}B \xrightarrow{\beta\otimes^{D}\mathsf{id}_B} B\otimes^{D}B \xrightarrow{m} B\ .$$

Let us check the axioms for a natural cotransformation in detail. As explained above, the commutativity of the first two diagrams in 4.2.2 is equivalent to the fact that $\beta \circ \alpha : {}_{\varphi}D \to {}_{\rho}B$ be a morphism of C-D-bicomodules. To see that this is the case we make explicit the bicomodule structures involved in the definition of $\beta \circ \alpha$ as a composition:

$${}_{\varphi}D \xrightarrow{\alpha} {}_{\psi}B \xrightarrow{t} {}_{\psi}D\otimes^{D}B \xrightarrow{\beta\otimes^{D}\mathsf{id}_B} {}_{\rho}B\otimes^{D}B \xrightarrow{m} {}_{\rho}B\ .$$

Since m, α, β and t are all morphisms of C-D-bicomodules as indicated, so is $\beta \circ \alpha$. The remaining condition is the commutativity of the diagram below, which holds by

(1) associativity of m

(2) conaturality of β

(3) conaturality of α

$$\begin{tikzcd}
 & & & A\otimes^{C}B \arrow[r, "\cong"] & A\otimes^{C}D\otimes^{D}B \arrow[r, "\rho_1\otimes^{D}\mathsf{id}_B"] & B\otimes^{D}B \arrow[dddr, "m"] & \\
 & & A\otimes^{C}B\otimes^{D}B \arrow[ur, "\mathsf{id}_A\otimes^{C}m"] \arrow[r, "\cong"] & A\otimes^{C}D\otimes^{D}B\otimes^{D}B \arrow[r, "\rho_1\otimes^{D}\mathsf{id}"] & B\otimes^{D}B\otimes^{D}B \arrow[ur, "\mathsf{id}_B\otimes^{D}m"] \arrow[dr, "m\otimes^{D}\mathsf{id}_B"] & (1) & \\
 & A\otimes^{C}D\otimes^{D}B \arrow[ur, "\mathsf{id}_A\otimes^{C}\beta\otimes^{D}\mathsf{id}_B"] \arrow[dr, "\psi_1\otimes^{D}\mathsf{id}_B"] & & (2) & & B\otimes^{D}B \arrow[dr, "m"] & \\
A\otimes^{C}B \arrow[ur, "\cong"] & & B\otimes^{D}B \arrow[r, "\cong"] \arrow[dr, "m"] & D\otimes^{D}B\otimes^{D}B \arrow[r, "\beta\otimes^{D}\mathsf{id}"] & B\otimes^{D}B\otimes^{D}B \arrow[ur, "m\otimes^{D}\mathsf{id}_B"] \arrow[dr, "\mathsf{id}_B\otimes^{D}m"] & (1) & B \\
A\otimes^{C}D \arrow[u, "\mathsf{id}_A\otimes^{C}\alpha"] \arrow[dd, "\varphi_1"] & (3) & & B \arrow[r, "\cong"] & D\otimes^{D}B \arrow[r, "\beta\otimes^{D}\mathsf{id}_B"] & B\otimes^{D}B \arrow[ur, "m"] & \\
 & & & & & (1) & \\
B \arrow[r, "\cong"'] & D\otimes^{D}B \arrow[r, "\alpha\otimes^{D}\mathsf{id}_B"'] & B\otimes^{D}B \arrow[r, "\cong"'] \arrow[uur, "m"] & D\otimes^{D}B\otimes^{D}B \arrow[r, "\beta\otimes^{D}\mathsf{id}"'] & B\otimes^{D}B\otimes^{D}B \arrow[r, "m\otimes^{D}\mathsf{id}_B"'] \arrow[uur, "\mathsf{id}_B\otimes^{D}m"] & B\otimes^{D}B \arrow[uuur, "m"'] &
\end{tikzcd}$$

Moreover, composition of natural cotransformations is associative, and unital with respect to the identity defined above. In fact, what we just did is to show that the set of all natural cotransformations $\{\alpha : \varphi \to \psi \ / \ \varphi, \psi : \mathfrak{C} \to \mathfrak{D}$ are cofunctors $\}$ is a submonoid of the monoid of admissible sections $\Gamma(\mathfrak{D})$, to be introduced in section 5.1.

One can similarly define the horizontal composition of a natural cotransformation with a cofunctor. One obtains in this way a 2-category of internal categories, cofunctors and natural cotransformations, that we still denote by $\overleftarrow{\boldsymbol{\mathit{Cat}}}_{\mathcal{S}}$.

Finally, let us mention that if the cofunctors φ and ψ are both the identity on objects, so that they can equivalently be seen as functors, then any natural cotransformation $\varphi \Rightarrow \psi$ can equivalently be seen as a natural transformation $\varphi \Rightarrow \psi$.

4.3 Functors versus cofunctors

In this section we compare the definitions of functors and cofunctors in various ways. These results should be useful in gaining familiarity with the a priori not so natural notion of cofunctor.

First, in the case $\mathcal{S} = \boldsymbol{\mathit{Sets}}$, we may describe functors $f : \mathfrak{C} \to \mathfrak{D}$ and cofunctors $\varphi : \mathfrak{C} \to \mathfrak{D}$ through pictures as follows.

$$\begin{tikzcd}
\mathfrak{C} \arrow[d, "f"'] & x \arrow[rr, bend left, "a"] \arrow[d, mapsto, "f_0"'] & {} \arrow[d, mapsto, "f_1"'] & x' \arrow[d, mapsto, "f_0"] \\
\mathfrak{D} & f_0(x) \arrow[rr, bend left, "f_1(a)"] & {} & f_0(x')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{D} & y \arrow[rr, bend left, "\varphi_1(a,y)"] \arrow[d, mapsto, "\varphi_0"'] & {} & y' \arrow[d, mapsto, "\varphi_0"] \\
\mathfrak{C} \arrow[u, "\varphi"] & \varphi_0(y) \arrow[rr, bend left, "a"'] & {} \arrow[u, mapsto, "\varphi_1"'] & \varphi_0(y')
\end{tikzcd}$$

These are meant to indicate the behavior of the various maps with respect to source and targets. Similar pictures can be used to describe compositions, unitality and associativity both for functors and cofunctors. We see that, just as a functor can be thought ot as "push-forward" of arrows from the category $\mathfrak{C}$ to the category $\mathfrak{D}$, a cofunctor may be thought of as a "lifting" of arrows from $\mathfrak{C}$ to $\mathfrak{D}$. Moreover, these push-forward and lifting should preserve identities and compositions in the obvious ways.

This model becomes even more meaningful in the following particular example. Let X and Y be topological spaces, $f : X \to Y$ a continuous map and $p : Y \to X$ a covering space map. Let $\pi(X)$ and $\pi(Y)$ denote the fundamental groupoids of X and Y. Then there is a functor $f_* : \pi(X) \to \pi(Y)$, obtained by pushing paths on X forward to Y through composition with f, but also a cofunctor $p^* : \pi(X) \to \pi(Y)$, obtained by lifting paths from X to Y along p (the unique lifting property of p guarantees that compositions and identities are preserved).

Next, we discuss an alternative description of the notion of functor, which highlights its "duality" with the notion of cofunctor. This is valid for any $\mathcal{S}$, and is based on the adjunction

$$\mathsf{Hom}_{C\text{–}D}(M, C_f{\otimes}^D N) \cong \mathsf{Hom}_{D\text{–}D}({}_fM, N) \ .$$

for C-D-bicomodules M and D-D-bicomodules N, where there is given a morphism of comonoids $f : C \to D$ (remark 3.0.1).

Let $\mathfrak{C} = (A, C, \dots)$ and $\mathfrak{D} = (B, D, \dots)$ be categories in $\mathcal{S}$ and $f : \mathfrak{C} \to \mathfrak{D}$ a functor, $f = (f_1, f_0)$ as in definition 4.1.1. Recall that $f_1 : {}_fA_f \to B$ is a morphism of D-D-bicomodules, where the subindices denote corestriction along f_0. Hence f_1 corresponds, under the adjunction above ($M = A_f$, $N = B$), to a morphism $\tilde{f}_1 : A_f \to C_f{\otimes}^D B$ of C-D-bicomodules. Let us rename

$$\varphi_0 = f_0 : C \to D \ \text{ and } \ \varphi_1 = \tilde{f}_1 : A_f \to C_f{\otimes}^D B \ .$$

One checks easily that the conditions in definition 4.1.1 for (f_1, f_0) translate into the following conditions for (φ_1, φ_0):

- $\varphi_0 : C \to D$ is a morphism of comonoids in $\mathcal{S}$,

- $\varphi_1 : A \to C \otimes^D B$ is a morphism of C-D-bicomodules in $\mathcal{S}$ (where A and C are viewed as right D-comodules by corestriction via φ_0),

and the following diagrams commute

$$\begin{tikzcd}
A \arrow[r, "\varphi_1"] & C \otimes^D B & A \otimes^C A \arrow[r, "\mathsf{id}_A \otimes^C \varphi_1"] \arrow[d, "m"'] & A \otimes^C C \otimes^D B \cong A \otimes^D B \arrow[r, "\varphi_1 \otimes^D \mathsf{id}_B"] & C \otimes^D B \otimes^D B \arrow[d, "\mathsf{id}_C \otimes^D m"] \\
C \arrow[u, "i"] \arrow[r, "\cong"] & C \otimes^D D \arrow[u, "\mathsf{id}_C \otimes^D i"'] & A \arrow[rr, "\varphi_1"'] & & C \otimes^D B
\end{tikzcd} .$$

These conditions are to be compared with those in definition 4.2.1 of cofunctor. The "duality" is remarkable.

There is a parallel alternative description of natural transformations, which bears the same dual relationship to that of natural cotransformations. Let $f, g : \mathfrak{C} \to \mathfrak{D}$ be functors and $\alpha : f \Rightarrow g$ a natural transformation as in definition 4.1.2. View f and g as pairs (φ_1, φ_0) and (ψ_1, ψ_0) as above. Recall that $\alpha : {}_{\psi}C_{\varphi} \to B$ is a morphism of D-D-bicomodules, where the subindices denote corestriction along $\varphi_0 = f_0$ and $\psi_0 = g_0$. Hence α corresponds, under the adjunction above ($M = C_{\varphi}$, $N = B$), to a morphism $\tilde{\alpha} : C_{\varphi} \to C_{\psi} \otimes^D B$ of C-D-bicomodules. Again, one checks easily that the conditions in definition 4.1.2 for α translate into the following conditions for $\tilde{\alpha}$: $\tilde{\alpha} : C_{\varphi} \to C_{\psi} \otimes^D B$ is a morphism of C-D-bicomodules and the following diagram commutes

$$\begin{tikzcd}
 & A \otimes^C C_{\psi} \otimes^D B \arrow[r, "\cong"] & A_{\psi} \otimes^D B \arrow[r, "\psi_1 \otimes^D \mathsf{id}_B"] & C_{\psi} \otimes^D B \otimes^D B \arrow[dr, "\mathsf{id}_C \otimes^D m"] & \\
A \otimes^C C_{\varphi} \arrow[ur, "\mathsf{id}_A \otimes^C \tilde{\alpha}"] \arrow[dr, "\cong"'] & & & & C_{\psi} \otimes^D B \\
 & A_{\varphi} \arrow[r, "\varphi_1"'] & C_{\varphi} \otimes^D B \arrow[r, "\tilde{\alpha} \otimes^D \mathsf{id}_B"'] & C_{\psi} \otimes^D B \otimes^D B \arrow[ur, "\mathsf{id}_C \otimes^D m"'] &
\end{tikzcd}$$

These conditions are to be compared with those in definition 4.2.2 of natural co-transformation.

Let us close this section by mentioning that both functors and cofunctors $\mathfrak{C} \to \mathfrak{D}$ naturally give rise to particular $\mathfrak{C}$-$\mathfrak{D}$-*birepresentations*. When $\mathcal{S} = \mathit{Sets}$, birepresentations are also called *profunctors* in the literature. For reasons of space, birepresentations will not be discussed in this work.

4.4 Cofunctors in *Sets*

Higgins and Mackenzie [HM] have introduced the notion of *comorphims* for Lie groupoids, and hence for ordinary small categories, by forgetting the additional structure (a Lie groupoid is after all a special type of small category). It is the purpose of this section to check that in the case $\mathcal{S} = \mathit{Sets}$, a cofunctor as in definition 4.2.1 is just a comorphism in the sense of [HM]. This section is otherwise independent of the rest of this work.

We start by recalling a construction for ordinary categories, sometimes known as the *Bousfield-Kan construction* or *homotopy colimit* [Lod2, appendix B.13]. Let $\mathfrak{C} = (A, C, \dots)$ be a category in Sets and $p : D \to C$ a map, where D is another set. Suppose that in addition there is given an *action of* $\mathfrak{C}$ *on* p, that is, for every arrow $a : x \to x'$ in $\mathfrak{C}$, there is given a map $\tilde{a} : p^{-1}(x) \to p^{-1}(x')$, and this assignment preserves identities and compositions. In this setting, the Bousfield-Kan construction is the category $\mathfrak{C} \ltimes D = (A \times^C D, D, \dots)$ where $A \times^C D = \{(a, y) \in A \times D \; /s(a) = p(y)\}$, a pair $(a, y) \in A \times^C D$ is an arrow from y to $\tilde{a}(y)$, and identities and composition are defined in the obvious way, so that the assignments $p : D \to C$ and $A \times^C D \to A, (a, y) \mapsto a$ define a functor $\mathfrak{C} \ltimes D \to \mathfrak{C}$. This can be illustrated as follows:

$$\begin{tikzcd}
\mathfrak{C} \ltimes D \arrow[d] & y \arrow[r, bend left=10, "{(a,y)}"] \arrow[d, mapsto, "p"'] & \tilde{a}(y) \arrow[d, mapsto, "p"] & . \\
\mathfrak{C} & x \arrow[r, bend left=10, "a"] & x' &
\end{tikzcd}$$

Now we are in position to state Higgins and Mackenzie's definition of comorphism. Let $\mathfrak{C} = (A, C, \dots)$ and $\mathfrak{D} = (B, D, \dots)$ be categories in ***Sets***. These authors define a comorphism $\mathfrak{C} \to \mathfrak{D}$ to consist of a map $p : D \to C$, an action of $\mathfrak{C}$ on p as above, and a functor $\mathfrak{C} \ltimes D \to \mathfrak{D}$ which is the identity on D (the objects).

This notion is equivalent to that of cofunctor, as we now explain. Given a cofunctor (φ_1, φ_0) as in definition 4.2.1, let $p = \varphi_0$, and define an action on $\mathfrak{C}$ on p as

$$\tilde{a}(y) = \tilde{t}(\varphi_1(a, y)) \in D \text{ for } (a, y) \in A \times^C D,$$

where $\tilde{t}$ is the target map of the category $\mathfrak{D}$. These data, together with the functor $(\varphi_1, \mathsf{id}_D) : \mathfrak{C} \ltimes D \to \mathfrak{D}$, define a comorphism in the sense of [HM]. It is clear how to proceed conversely.

This description of cofunctors in terms of actions holds true for arbitrary lex categories $\mathcal{S}$, not only for $\mathcal{S} =$ ***Sets***, but it does not seem possible to extend it to the case of general monoidal categories.

Chapter 5

The monoid of admissible sections

5.1 Definition

Definition 5.1.1. Let $\mathfrak{C} = (A, C, s, t, i, m)$ a category in $\mathcal{S}$. An admissible section for $\mathfrak{C}$ is a map $u : C \to A$ in $\mathcal{S}$ such that

$$\begin{tikzcd}
C \arrow[r, "u"] \arrow[d, "\Delta_C"'] & A \arrow[d, "s"] \\
C \otimes C \arrow[r, "u \otimes \mathsf{id}_C"'] & A \otimes C
\end{tikzcd}$$

commutes. The set of admissible sections for $\mathfrak{C}$ is denoted by $\Gamma(\mathfrak{C})$.

In other words, an admissible section is a morphism of right C-comodules $u : C \to A$. An admissible section of an ordinary small category (in $\mathtt{Sets}$) $\mathfrak{C} = (A, X, \tilde{s}, \tilde{t}, \dots)$ is just a map $u : X \to A$ such that $\tilde{s}u(c) = c \; \forall x \in X$:

$$\begin{tikzcd}
x \arrow[r, bend left, "u(x)"] & \bullet .
\end{tikzcd}$$

Admissible sections of ordinary categories were introduced by Chase in the context of affine groupoid schemes [Cha1]. They have also been considered in the Lie groupoid literature [Mac, definition II.5.1]; in this context the notion apparently goes back to Ehresmann.

In the proof of the following result we will make use of every single axiom in the definition of internal categories 2.3.1.

Proposition 5.1.1. *Let* $\mathfrak{C}$ *be as before. Then* $\Gamma(\mathfrak{C})$ *is an ordinary monoid as follows:*

- *the unit element is* $i : C \to A$*, and*
- *given* u *and* v *in* $\Gamma(\mathfrak{C})$*, their product is*

$$u * v : C \xrightarrow{v} A \xrightarrow{t} C{\otimes^C}A \xrightarrow{u\otimes^C\mathsf{id}_A} A{\otimes^C}A \xrightarrow{m} A\ .$$

Proof. First notice that indeed $u * v \in \Gamma(\mathfrak{C})$ and $i \in \Gamma(\mathfrak{C})$, because all v, t, id_A, m and i are morphisms of right C-comodules; the latter two by definition of internal categories.

Also, i is a morphism of left C-comodules, so $t \circ i = (\mathsf{id}_C{\otimes}i) \circ \Delta_C$, hence

$$u * i = m \circ (u{\otimes}\mathsf{id}_A) \circ t \circ i = m \circ (u{\otimes}\mathsf{id}_A) \circ (\mathsf{id}_C{\otimes}i) \circ \Delta_C$$
$$= m \circ (\mathsf{id}_A{\otimes}i) \circ (u{\otimes}\mathsf{id}_C) \circ \Delta_C = m \circ (\mathsf{id}_A{\otimes}i) \circ s \circ u = u,$$

by right unitality for m and i, and since u is a morphism of right C-comodules. On the other hand, left unitality for i and m yields directly that $i * v = v$.

Finally, associativity for $*$ is the commutativity of the following diagram (the top boundary is $(u*v)*w$, the bottom $u*(v*w)$), which holds because m preserves targets (is a morphism of left C-comodules) and is associative:

$$\begin{tikzcd}
 & A \arrow[r, "t"] & C\otimes^C A \arrow[r, "v\otimes^C \mathsf{id}_A"] & A\otimes^C A \arrow[r, "t\otimes^C \mathsf{id}_A"] & C\otimes^C A\otimes^C A \arrow[r, "u\otimes^C \mathsf{id}_A\otimes^C \mathsf{id}_A"] \arrow[dr, "\mathsf{id}_C\otimes^C m"] & A\otimes^C A\otimes^C A \arrow[r, "m\otimes^C \mathsf{id}_A"] \arrow[dr, "\mathsf{id}_A\otimes^C m"] & A\otimes^C A \arrow[dr, "m"] & \\
C \arrow[ur, "w"] \arrow[dr, "w"'] & & & & & & & A \\
 & A \arrow[r, "t"'] & C\otimes^C A \arrow[r, "v\otimes^C \mathsf{id}_A"'] & A\otimes^C A \arrow[r, "m"'] \arrow[uur, "t\otimes^C \mathsf{id}_A"] & A \arrow[r, "t"'] & C\otimes^C A \arrow[r, "u\otimes^C \mathsf{id}_A"'] & A\otimes^C A \arrow[ur, "m"'] &
\end{tikzcd}$$

□

When $\mathcal{S} = \texttt{\textit{Sets}}$, multiplication of admissible sections can be described through the following simple picture:

$$\begin{tikzcd}
x \arrow[r, bend left, "v(x)"] \arrow[rr, bend right, "(u*v)(x)"'] & \tilde{t}v(x) \arrow[r, bend left, "u(\tilde{t}v(x))"] & \bullet
\end{tikzcd}$$

5.2 Examples

We first compute the monoid of admissible sections for the discrete, one-object and pair categories (section 2.4).

For the discrete category $\widehat{\mathrm{C}}$ we have, by remark 3.0.1,

$$\Gamma(\widehat{\mathrm{C}}) = \mathsf{Hom}_C(C, C) \cong \mathsf{Hom}_{\mathcal{S}}(C, I) \text{ under } u \mapsto \epsilon_C \circ u \ .$$

Let us check that this is actually an isomorphism of monoids $\Gamma(\widehat{\mathrm{C}}) \cong \mathsf{Hom}_{\mathcal{S}}(C, I)^{op}$. Here, $\mathsf{Hom}_{\mathcal{S}}(C, I)$ is a monoid under convolution, which we also denote by the symbol

$*$: $f * g = (f \otimes g) \circ \Delta_C$. First, the unit element $i = \mathsf{id}_C \in \Gamma(\widehat{\mathrm{C}})$ maps to the unit element $\epsilon_C \in \mathsf{Hom}_{\mathcal{S}}(C, I)$. Second, the fact that multiplication of admissible sections corresponds to the opposite of convolution is the commutativity of the following diagram, where the top boundary is $\epsilon_C \circ (u * v)$ and the bottom $(\epsilon_C \circ v) * (\epsilon_C \circ u)$:

$$\begin{tikzcd}
C \arrow[r, "v"] \arrow[d, "\Delta_C"'] & C \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"] & C \otimes^C C \arrow[r, "u \otimes^C \mathsf{id}_C"] \arrow[d, "\mathsf{id}_C \otimes \epsilon_C"'] & C \otimes^C C \arrow[r, "\Delta_C^{-1}"] \arrow[d, "\epsilon_C \otimes \epsilon_C"] & C \arrow[d, "\epsilon_C"] \\
C \otimes C \arrow[r, "v \otimes \mathsf{id}_C"'] \arrow[rrd, bend right, "(\epsilon_C \circ v) \otimes \mathsf{id}_C"'] & C \otimes C \arrow[rd, "\epsilon_C \otimes \mathsf{id}_C"] & C \otimes I \arrow[r, "(\epsilon_C \circ u) \otimes \mathsf{id}_I"'] \arrow[d, equal] & I \otimes I \arrow[r, equal] & I \\
 & & I \otimes C \arrow[ru, bend right, "\mathsf{id}_I \otimes (\epsilon_C \circ u)"'] & &
\end{tikzcd}$$

(The first square commutes by definition of admissible section: v is a morphism of right C-comodules).

For the one-object category $\underset{\vee}{\mathrm{A}}$ we have, since $\mathsf{Comod}_{\mathcal{S}}(I) = \mathcal{S}$,

$$\Gamma(\underset{\vee}{\mathrm{A}}) = \mathsf{Hom}_I(I, A) = \mathsf{Hom}_{\mathcal{S}}(I, A).$$

It is immediate from the definition that multiplication of admissible sections corresponds to the convolution product in $\mathsf{Hom}_{\mathcal{S}}(I, A)$.

For the pair category $\widehat{\underset{\vee}{\mathrm{C}}}$, we can use remark 3.0.1 again to conclude that

$$\Gamma(\widehat{\underset{\vee}{\mathrm{C}}}) = \mathsf{Hom}_C(C, C \otimes C) \cong \mathsf{End}_{\mathcal{S}}(C) \text{ under } u \mapsto (\mathsf{id}_C \otimes \epsilon_C) \circ u\ .$$

Let us check that this is actually an isomorphism of monoids, where we view $\mathsf{End}_{\mathcal{S}}(C)$ as a monoid under composition. In fact, the unit element $i = \Delta_C \in \Gamma(\widehat{\underset{\vee}{\mathrm{C}}})$ maps by counitality to $(\mathsf{id}_C \otimes \epsilon_C) \circ \Delta_C = \mathsf{id}_C$, the unit element of $\mathsf{End}_{\mathcal{S}}(C)$. The fact that multiplication of admissible sections corresponds to composition of maps is the commutativity of the following diagram, where the top boundary is $(\mathsf{id}_C \otimes \epsilon_C) \circ (u * v)$ and the bottom $((\mathsf{id}_C \otimes \epsilon_C) \circ u) \circ ((\mathsf{id}_C \otimes \epsilon_C) \circ v)$:

$$\begin{tikzcd}
C \arrow[r, "v"] \arrow[dr, bend right, "(\mathsf{id}_C \otimes \epsilon_C) \circ v"'] & C \otimes C \arrow[r, "\Delta_C \otimes \mathsf{id}_C"] \arrow[d, "\mathsf{id}_C \otimes \epsilon_C"'] \arrow[dr, "u \otimes \mathsf{id}_C"] & C \otimes C \otimes C \arrow[r, "u \otimes \mathsf{id}_{C \otimes C}"] & C \otimes C \otimes C \otimes C \arrow[r, "\mathsf{id}_C \otimes \epsilon_C \otimes \epsilon_C \otimes \mathsf{id}_C"] & C \otimes C \arrow[d, "\mathsf{id}_C \otimes \epsilon_C"] \\
 & C \arrow[rrr, bend right, "(\mathsf{id}_C \otimes \epsilon_C) \circ u"'] & C \otimes C \otimes C \arrow[ur, "\mathsf{id}_C \otimes \Delta_C \otimes \mathsf{id}_C"] \arrow[rr, "\mathsf{id}_C \otimes \epsilon_C \otimes \epsilon_C"'] & & C
\end{tikzcd}$$

(The triangle commutes by definition of admissible section: u is a morphism of right C-comodules).

For the empty category Φ_C (assuming $\mathcal{S}$ has a zero object), obviously $\Gamma(\Phi_C) = \{0\}$, the trivial monoid.

We close the section by announcing two important results that will be proved in later chapters.

We have mentioned that a category in ***Groups*** can be equivalently described as a crossed module of groups. The monoid of admissible sections of such a category turns out to be Whitehead's monoid of derivations of the crossed module [W, N]. This will be proved in section 10.1.

The monoid of admissible sections of a linear category (which, as already said, is a particular example of a category in $\boldsymbol{\mathit{Vec}}_k$) coincides (essentially) with Mitchell's matrix ring of the linear category [Mit]. This will be proved in section 9.1. In particular Rota's incidence algebra of a locally finite poset [Rot] is an instance of this.

5.3 Cofunctors and admissible sections

As announced in the introduction, the construction of admissible sections is functorial with respect to cofunctors, this being the main reason for our interest in this type of morphisms for internal categories.

Let $\mathfrak{C} = (A, C, \dots)$ and $\mathfrak{D} = (B, D, \dots)$ be categories in $\mathcal{S}$ and $\varphi : \mathfrak{C} \to \mathfrak{D}$ a cofunctor. Given $u : C \to A$ in $\Gamma(\mathfrak{C})$, define $\Gamma(\varphi)(u) : D \to B$ as the composite

$$\Gamma(\varphi)(u) : D \xrightarrow{\Delta_D} D \otimes^{D} D \hookrightarrow D_{\varphi}\otimes^{C}{}_{\varphi}D \xrightarrow{\varphi_0 \otimes^{C} \mathsf{id}_D} C \otimes^{C}{}_{\varphi}D \xrightarrow{u \otimes^{C} \mathsf{id}_D} A \otimes^{C}{}_{\varphi}D \xrightarrow{\varphi_1} B \ ,$$

where the subindices denote corestriction via $\varphi_0 : D \to C$ as in section 3, and the canonical map of lemma 3.0.1 has been used. Notice that since φ_1 is a morphism of right D-comodules, so is $\Gamma(\varphi)(u)$, i.e. $\Gamma(\varphi)(u) \in \Gamma(\mathfrak{D})$.

When $\mathcal{S} = \mathtt{Sets}$, this definition can be illustrated as follows. Given $u \in \Gamma(\mathfrak{C})$ and y an object of $\mathfrak{D}$, $\Gamma(\varphi)(u)(y)$ is the lift of $u(\varphi_0(y))$ to y provided by φ_1:

$$\begin{tikzcd}
\mathfrak{D} & y \arrow[d, mapsto, "\varphi_0"'] \arrow[rr, dashed, bend left, "\Gamma(\varphi)(u)(y)", ""{name=U, below}] & & \bullet \\
\mathfrak{C} \arrow[u, "\varphi"] & \varphi_0(y) \arrow[rr, bend left, "u(\varphi_0(y))"{name=L}] & & \bullet
\arrow[from=L, to=U, mapsto, "\varphi_1"']
\end{tikzcd}$$

Proposition 5.3.1. $\Gamma(\varphi) : \Gamma(\mathfrak{C}) \to \Gamma(\mathfrak{D})$ *is a morphism of monoids.*

Proof. First, $\Gamma(\varphi)(i) = i$ precisely by unitality for the cofunctor φ. Second, the fact that $\Gamma(\varphi)(u * v) = \Gamma(\varphi)(u) * \Gamma(\varphi)(v)$ is the commutativity of the following diagram (where $\Gamma(\varphi)(u * v)$ is the top boundary and $\Gamma(\varphi)(u) * \Gamma(\varphi)(v)$ the bottom one), which holds by

(1) φ_1 is a morphism of left C-comodules

(2) associativity for the cofunctor φ

( ) the unlabeled diagrams commute by naturality or functoriality

$$\begin{tikzcd}
 & & & & C\otimes^{C}A\otimes^{C}D \arrow[r, "u\otimes^{C}\mathsf{id}_A\otimes^{C}\mathsf{id}_D"] \arrow[d, "\mathsf{id}_C\otimes^{C}\varphi_1"] & A\otimes^{C}A\otimes^{C}D \arrow[r, "m\otimes^{C}\mathsf{id}_D"] \arrow[d, "\mathsf{id}_A\otimes^{C}\varphi_1"] & A\otimes^{C}D \arrow[dr, "\varphi_1"] & \\
D \arrow[r, "\cong"] & C\otimes^{C}D \arrow[r, "v\otimes^{C}\mathsf{id}_D"] & A\otimes^{C}D \arrow[urr, "t\otimes^{C}\mathsf{id}_D"] \arrow[dl, "\varphi_1"'] & (1) & C\otimes^{C}B \arrow[r, "u\otimes^{C}\mathsf{id}_B"] & A\otimes^{C}B \arrow[d, "\cong"] & (2) & B \\
 & B \arrow[r, "t"'] & D\otimes^{D}B \arrow[urr, "\cong"] \arrow[rr, "\cong"'] & & C\otimes^{C}D\otimes^{D}B \arrow[r, "u\otimes^{C}\mathsf{id}_D\otimes^{D}\mathsf{id}_B"'] & A\otimes^{C}D\otimes^{D}B \arrow[r, "\varphi_1\otimes^{D}\mathsf{id}_B"'] & B\otimes^{D}B \arrow[ur, "m"'] &
\end{tikzcd}$$

(In this diagram, D-comodules are viewed, when convenient, as C-comodules by corestriction via φ_0). ☐

The functoriality of Γ is easier. First, $\Gamma(\mathsf{id}_{\mathfrak{C}})(u)$ is by definition the following composite

$$C \xrightarrow{\Delta_C} C\otimes^{C}C \xrightarrow{u\otimes^{C}\mathsf{id}_C} A\otimes^{C}C \xrightarrow{\cong} A$$

which is just $u : C \to A$. Second, given cofunctors $\varphi : \mathfrak{C} \to \mathfrak{D}$ and $\psi : \mathfrak{D} \to \mathfrak{E}$ as in section 4.2, $\Gamma(\psi \circ \varphi)(u)$ is, by definition, the following composite

$$E \xrightarrow{\Delta_E} E\otimes^{E}E \xrightarrow{\psi_0\varphi_0\otimes^{C}\mathsf{id}_E} C\otimes^{C}E \xrightarrow{u\otimes^{C}\mathsf{id}_D} P\otimes^{C}D \xrightarrow{\cong} P\otimes^{C}D\otimes^{D}E \xrightarrow{\varphi_1\otimes^{D}\mathsf{id}_E} Q\otimes^{D}E \xrightarrow{\psi_1} R$$

which clearly coincides with $\Gamma(\psi)\Gamma(\varphi)(u)$.

The above shows that admissible sections is a functor

$$\Gamma : \overleftarrow{\mathit{Cat}}_{\mathbb{S}} \to \mathit{Monoids} \ .$$

Actually, more than this holds: the 2-categorical structure of $\overleftarrow{\mathit{Cat}}_{\mathbb{S}}$ is also preserved, in the following sense.

Proposition 5.3.2. *Let $\alpha : \varphi \Rightarrow \psi$ be a natural cotransformation between two cofunctors as above. Then $\alpha \in \Gamma(\mathfrak{D})$, and for any $u \in \Gamma(\mathfrak{C})$,*

$$\alpha * \Gamma(\varphi)(u) = \Gamma(\psi)(u) * \alpha \ .$$

Proof. By definition of natural cotransformation (first diagram in 4.2.2), α is a morphism of right D-comodules, i.e. $\alpha \in \Gamma(\mathfrak{D})$. The equality above is the commutativity of the diagram below (where the top boundary is $\alpha * \Gamma(\varphi)(u)$ and the bottom $\Gamma(\psi)(u) * \alpha$), where

( ) the unlabeled diagrams commute by naturality or functoriality

(1) second diagram in def. 4.2.2 for α

(2) third diagram in def. 4.2.2 for α

(3) commutes because it does after composing with the isomorphism $C\otimes^{C}D\otimes^{D}B \xrightarrow{\mathsf{id}_C\otimes^{C}(\epsilon_D\otimes\mathsf{id}_B)} C\otimes^{C}B$, by counitality of t and Δ_D.

$$\begin{tikzcd}
{}_{\varphi}D \arrow[r, "\Delta_D"] \arrow[dd, "\alpha"'] & D\otimes^{D}D \arrow[r, "\varphi_0\otimes\mathsf{id}_D"] & C\otimes^{C}{}_{\varphi}D \arrow[r, "u\otimes^{C}\mathsf{id}_D"] \arrow[d, "\mathsf{id}_C\otimes^{C}\alpha"'] & A\otimes^{C}{}_{\varphi}D \arrow[r, "\varphi_1"] \arrow[d, "\mathsf{id}_A\otimes^{C}\alpha"] & B \arrow[r, "t"] & D\otimes^{D}B \arrow[r, "\alpha\otimes^{D}\mathsf{id}_B"] & B\otimes^{D}B \arrow[d, "m"] \\
 & (1) & C\otimes^{C}{}_{\psi}B \arrow[r, "u\otimes^{C}\mathsf{id}_B"] \arrow[dr, "\mathsf{id}_C\otimes^{C}t"] \arrow[d, phantom, "(3)"] & A\otimes^{C}{}_{\psi}B \arrow[dr, "\mathsf{id}_A\otimes^{C}t"] & & (2) & B \\
{}_{\psi}B \arrow[r, "t"'] & D\otimes^{D}B \arrow[r, "\Delta_D\otimes^{D}\mathsf{id}_B"'] \arrow[ur, "\psi_0\otimes\mathsf{id}_B"] & D\otimes^{D}D\otimes^{D}B \arrow[r, "\psi_0\otimes\mathsf{id}_D\otimes^{D}\mathsf{id}_B"'] & C\otimes^{C}{}_{\psi}D\otimes^{D}B \arrow[r, "u\otimes^{C}\mathsf{id}_D\otimes^{D}\mathsf{id}_B"'] & A\otimes^{C}{}_{\psi}D\otimes^{D}B \arrow[rr, "\psi_1\otimes^{D}\mathsf{id}_B"'] & & B\otimes^{D}B \arrow[u, "m"']
\end{tikzcd}$$

□

Proposition 5.3.2 allows us to view natural contransformations as admissible sections. Notice that the composition of natural cotransformations defined in section

4.2 is then just a special case of multiplication of admissible sections. This had been announced in section 4.2.

We can now derive a result that was already mentioned in section 4.1, when computing the monoid of endomorphisms of the identity functor (examples 4.1.2).

Corollary 5.3.1. *For any category* $\mathfrak{C}$*,* $\mathsf{End}(\mathsf{id}_{\mathfrak{C}}) \subseteq Z(\Gamma(\mathfrak{C}))$*, where* $\mathsf{End}(\mathsf{id}_{\mathfrak{C}})$ *denotes the monoid of natural transformations* $\mathsf{id}_{\mathfrak{C}} \Rightarrow \mathsf{id}_{\mathfrak{C}}$*.*

Proof. Immediate from proposition 5.3.2, plus the observation that natural transformations coincide with natural cotransformations in this case (section 4.2). ☐

5.4 Functors and admissible sections

In general, a functor $f : \mathfrak{C} \to \mathfrak{D}$ between internal categories does not induce a morphism $\Gamma(\mathfrak{C}) \to \Gamma(\mathfrak{D})$ (unless, for instance, if it is the identity on objects, i.e. it can be seen as a cofunctor). For instance, consider the functor $\epsilon : \widehat{\mathrm{C}} \to \hat{\mathrm{I}}$ between discrete categories, and assume for simplicity that $\mathcal{S} = \mathsf{Vec}_k$. We know from section 5.2 that in this case $\Gamma(\widehat{\mathrm{C}}) = (C^*)^{op}$ and $\Gamma(\hat{\mathrm{I}}) = k$, so we certainly do not expect any morphism $\Gamma(\widehat{\mathrm{C}}) \to \Gamma(\hat{\mathrm{I}})$. On the other hand, in general a functor $f : \mathfrak{C} \to \mathfrak{D}$ does not induce a morphism $\Gamma(\mathfrak{D}) \to \Gamma(\mathfrak{C})$ either: consider the case of one-object categories.

However, it is possible to pull-back subsets of $\Gamma(\mathfrak{D})$ to $\Gamma(\mathfrak{C})$ along a functor $f : \mathfrak{C} \to \mathfrak{D}$, in a way that is compatible with the multiplication of admissible sections. For each $x \in \Gamma(\mathfrak{D})$ define

$$f^{-1}(x) = \{u \in \Gamma(\mathfrak{C}) \ / \ \begin{tikzcd} C \arrow[r, "u"] \arrow[d, "f_0"'] & A \arrow[d, "f_1"] \\ D \arrow[r, "x"'] & B \end{tikzcd} \quad \text{commutes } \}.$$

Proposition 5.4.1. *Let $f : \mathfrak{C} \to \mathfrak{D}$ be a functor as above.*

1. *For any $x, y \in \Gamma(\mathfrak{D})$, $f^{-1}(x) * f^{-1}(y) \subseteq f^{-1}(x * y)$.*

2. *$f^{-1}(i)$ is a submonoid of $\Gamma(\mathfrak{C})$.*

Proof.

1. For any $u \in f^{-1}(x)$ and $v \in f^{-1}(y)$, the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "v"] \arrow[d, "f_0"'] & A \arrow[r, "t"] \arrow[d, "f_1"'] & C \otimes^C A \arrow[r, "u \otimes^C \mathsf{id}_A"] \arrow[d, "f_0 \otimes^D f_1"] & A \otimes^C A \arrow[r, "m"] \arrow[d, "f_1 \otimes^D f_1"] & A \arrow[d, "f_1"] \\ D \arrow[r, "y"'] & B \arrow[r, "t"'] & D \otimes^D B \arrow[r, "x \otimes^D \mathsf{id}_B"'] & B \otimes^D B \arrow[r, "m"'] & B \end{tikzcd} \quad ,$$

   by definition of functor 4.1.1 and of f^{-1}. Hence $u * v \in f^{-1}(x * y)$.

2. It only remains to show that $i \in f^{-1}(i)$. This is the unitality condition for f in definition 4.1.1.

$\square$

More generally, for any subset $X \subseteq \Gamma(\mathfrak{D})$ one can define

$$f^{-1}(X) = \bigcup_{x \in X} f^{-1}(x) \subseteq \Gamma(\mathfrak{C}) \text{ and } f^{-1}(\emptyset) = \emptyset \ .$$

It follows from the proposition above that for any $X, Y \subseteq \Gamma(\mathfrak{D})$, $f^{-1}(X) * f^{-1}(Y) \subseteq f^{-1}(X * Y)$, and that if M is a submonoid of $\Gamma(\mathfrak{D})$ then $f^{-1}(M)$ is a submonoid of $\Gamma(\mathfrak{C})$. Thus:

Corollary 5.4.1. *A functor* $f : \mathfrak{C} \to \mathfrak{D}$ *induces an order-preserving mapping from the lattice of submonoids of* $\Gamma(\mathfrak{D})$ *to that of* $\Gamma(\mathfrak{C})$.

Proof. Done above. $\square$

Chapter 6

Representations of an internal category

6.1 Definition and examples

Definition 6.1.1. Let $\mathfrak{C} = (A, C, s, t, i, m)$ be a category in $\mathcal{S}$. A (left) representation of $\mathfrak{C}$ in $\mathcal{S}$ is an object X of $\mathcal{S}$ together with:

- a left C-comodule structure $p : X \to C \otimes X$ and
- a morphism of left C-comodules $a : A \otimes^C X \to X$,

such that the following diagrams commute

$$\begin{tikzcd}
A\otimes^C X \arrow[r, "a"] & X \arrow[dl, "\cong", "p"'] & A\otimes^C A\otimes^C X \arrow[r, "\mathsf{id}_A\otimes^C a"] \arrow[d, "m\otimes^C\mathsf{id}_X"'] & A\otimes^C X \arrow[d, "a"] \\
C\otimes^C X \arrow[u, "i\otimes^C\mathsf{id}_X"] & & A\otimes^C X \arrow[r, "a"'] & X
\end{tikzcd} .$$

Sometimes we refer to $a : A{\otimes^C}X \to X$ as the A-action on X, and to the diagrams above as the unitality and associativity of the action.

A morphism between representations X and Y of $\mathfrak{C}$ is a map $f : X \to Y$ in $\mathcal{S}$ such that both diagrams below commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p_X"'] & Y \arrow[d, "p_Y"] & A{\otimes^C}X \arrow[r, "\mathsf{id}_A{\otimes^C}f"] \arrow[d, "a_X"'] & A{\otimes^C}Y \arrow[d, "a_Y"] \\
C{\otimes}X \arrow[r, "\mathsf{id}_C{\otimes}f"'] & C{\otimes}Y & X \arrow[r, "f"'] & Y
\end{tikzcd} ;$$

in particular f is a morphism of left C-comodules. The resulting category $\mathsf{Rep}_{\mathcal{S}}\mathfrak{C}$ comes thus equipped with a forgetful functor $\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \to \mathsf{Comod}_{\mathcal{S}}C$.

Examples 6.1.1.

For any category $\mathfrak{C} = (A, C, \ldots)$, A itself is a representation, via $t : A \to C{\otimes}A$ and $m : A{\otimes^C}A \to A$. This is called the (left) *regular* representation of $\mathfrak{C}$.

Obviously, a representation of a one-object category is just a module over the underlying monoid: $\mathsf{Rep}_{\mathcal{S}}(\underset{\vee}{\mathrm{A}}) = \mathsf{Mod}_{\mathcal{S}}A$. Also, a representation of a discrete category is just a comodule over the underlying comonoid: $\mathsf{Rep}_{\mathcal{S}}(\widehat{\mathrm{C}}) = \mathsf{Comod}_{\mathcal{S}}C$ (unitality forces $a = p^{-1}$).

When $\mathcal{S}$ is a lex category, the definition above coincides with the definition of *internal diagram* or *internal functor* for usual internal categories [Joh, definition 2.14]. The reason for this terminology is that, specializing even further to the case when $\mathfrak{C}$ is a category in $\mathcal{S} = \mathtt{Sets}$, a representation of $\mathfrak{C}$ is just a functor $\mathfrak{C} \to \mathtt{Sets}$. In fact, given a functor $F : \mathfrak{C} \to \mathtt{Sets}$, one obtains a representation X_F of $\mathfrak{C}$ as follows:

$$X_F = \coprod_{c \in C} F(c) \ ,$$

with the C-comodule structure corresponding to the given C-grading (examples 1.4.1), and with A-action $A \times^C X_F \to X_F$, $(a, x) \mapsto F(a)(x)$. Conversely, any representation arises from a unique functor in this way. Also, morphisms of representations correspond to natural transformations of functors.

Let us remark the obvious fact that, in the general case (arbitrary $\mathcal{S}$), it does not make sense to talk about functors $\mathfrak{C} \to \mathit{Sets}$ or $\mathfrak{C} \to \mathcal{S}$.

6.2 Restriction along functors and cofunctors

Let $\mathfrak{C} = (A, C, \dots)$ and $\mathfrak{D} = (B, D, \dots)$ be categories in $\mathcal{S}$ and $\varphi = (\varphi_1, \varphi_0) : \mathfrak{C} \to \mathfrak{D}$ a cofunctor. We will define a functor (between ordinary large categories) $\mathsf{Rep}_{\mathcal{S}}\mathfrak{D} \to \mathsf{Rep}_{\mathcal{S}}\mathfrak{C}$. Let (X, p, a) be a representation of $\mathfrak{D}$. First, view X as a C-comodule by corestriction along φ_0 (section 3), that is via

$$\tilde{p} : X \xrightarrow{p} D \otimes X \xrightarrow{\varphi_0 \otimes \mathsf{id}_X} C \otimes X \ .$$

Then, consider the map

$$\tilde{a} : A \otimes^C X \cong A \otimes^C D \otimes^D X \xrightarrow{\varphi_1 \otimes^D \mathsf{id}_X} B \otimes^D X \xrightarrow{a} X \ .$$

Proposition 6.2.1. *The resulting $(X, \tilde{p}, \tilde{a})$ is a representation of $\mathfrak{C}$.*

Proof. $\tilde{a}$ is a morphism of left C-comodules, since so are φ_1 and a. We only need to check unitality and associativity for the A-action $\tilde{a}$. Unitality is the commutativity of the following diagram, which holds by

(1) unitality for φ

(2) unitality for a

$$\begin{tikzcd}
A\otimes^{C} X \arrow[r, "\cong"] & A\otimes^{C} D\otimes^{D} X \arrow[r, "\varphi_1\otimes^{D}\mathsf{id}_X"] & B\otimes^{D} X \arrow[r, "a"] & X \arrow[dl, bend left, "p"] \\
C\otimes^{C} X \arrow[u, "i\otimes^{C}\mathsf{id}_X"] \arrow[r, "\cong"'] & C\otimes^{C} D\otimes^{D} X \arrow[u, "i\otimes^{C}\mathsf{id}_D\otimes^{D}\mathsf{id}_X"] \arrow[ur, phantom, "(1)"] & D\otimes^{D} X \arrow[l, "\cong"] \arrow[u, "i\otimes^{D}\mathsf{id}_X"] \arrow[ur, phantom, "(2)"] &
\end{tikzcd} \quad .$$

(To see that the map $D\otimes^{D} X \to C\otimes^{C} X$ is induced by $\varphi_0\otimes\mathsf{id}_X$, compose with the isomorphism $C\otimes^{C} D\otimes^{D} X \xrightarrow{\mathsf{id}_C\otimes^{C}(\epsilon_D\otimes\mathsf{id}_X)} C\otimes^{C} X$ and use counitality of p and Δ_D).

Associativity is the commutativity of the following diagram, which holds by

(1) associativity for φ

(2) associativity for a

$$\begin{tikzcd}
A\otimes^{C} A\otimes^{C} X \arrow[r, "\cong"] \arrow[ddd, "m\otimes^{C}\mathsf{id}_X"'] & A\otimes^{C} A\otimes^{C} D\otimes^{D} X \arrow[r, "\mathsf{id}_A\otimes^{C}\varphi_1\otimes^{D}\mathsf{id}_X"] \arrow[ddd, "m\otimes^{C}\mathsf{id}_D\otimes^{D}\mathsf{id}_X"'] & A\otimes^{C} B\otimes^{D} X \arrow[r, "\mathsf{id}_A\otimes^{C} a"] \arrow[d, "\cong"] & A\otimes^{C} X \arrow[d, "\cong"] \\
 & & A\otimes^{C} D\otimes^{D} B\otimes^{D} X \arrow[d, "\varphi_1\otimes^{D}\mathsf{id}_B\otimes^{D}\mathsf{id}_X"] & A\otimes^{C} D\otimes^{D} X \arrow[d, "\varphi_1\otimes^{D}\mathsf{id}_X"] \\
 & (1) & B\otimes^{D} B\otimes^{D} X \arrow[r, "\mathsf{id}_B\otimes^{D} a"] \arrow[d, "m\otimes^{D}\mathsf{id}_X"] \arrow[dr, phantom, "(2)"] & B\otimes^{D} X \arrow[d, "a"] \\
A\otimes^{C} X \arrow[r, "\cong"'] & A\otimes^{C} D\otimes^{D} X \arrow[r, "\varphi_1\otimes^{D}\mathsf{id}_X"'] & B\otimes^{D} X \arrow[r, "a"'] & X
\end{tikzcd} \quad .$$

□

We sometimes write $\mathsf{res}_\varphi(X)$ for the resulting representation of $\mathfrak{C}$, and say that $\mathsf{res}_\varphi(X)$ has been obtained from X by restriction along φ. Clearly, a morphism $f : X \to Y$ of representations of $\mathfrak{D}$ is also a morphism between the corresponding representations of $\mathfrak{C}$. This defines a functor

$$\mathsf{res}_\varphi : \mathsf{Rep}_{\mathcal{S}}\mathfrak{D} \to \mathsf{Rep}_{\mathcal{S}}\mathfrak{C},$$

which by construction fits into a commutative diagram

$$\begin{tikzcd}
\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \arrow[r] & \mathsf{Comod}_{\mathcal{S}}C \\
\mathsf{Rep}_{\mathcal{S}}\mathfrak{D} \arrow[r] \arrow[u, "\mathsf{res}_\varphi"] & \mathsf{Comod}_{\mathcal{S}}D \arrow[u, "\mathsf{cores}_{\varphi_0}"']
\end{tikzcd},$$

where the unlabelled arrows are forgetful functors. Notice that res_φ preserves the forgetful functors to $\mathcal{S}$, i.e.

$$\begin{tikzcd}
\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \arrow[dr] & \\
 & \mathcal{S} \\
\mathsf{Rep}_{\mathcal{S}}\mathfrak{D} \arrow[uu, "\mathsf{res}_\varphi"] \arrow[ur] &
\end{tikzcd}$$

commutes. Incidentally, the forgetful functor $\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \to \mathcal{S}$ is $\mathsf{res}_{\overleftarrow{i}}$ where $\overleftarrow{i} : \mathfrak{I} \to \mathfrak{C}$ is the cofunctor of examples 4.2.1.

Now, let $\varphi, \psi : \mathfrak{C} \to \mathfrak{D}$ be two cofunctors and $\alpha : \varphi \Rightarrow \psi$ a natural cotransformation (definition 4.2.2). For each $\mathfrak{D}$-representation X, define

$$\alpha_X : {}_\varphi X \cong {}_\varphi D \otimes^D X \xrightarrow{\alpha \otimes^D \mathsf{id}_X} {}_\psi B \otimes^D X \xrightarrow{a} {}_\psi X,$$

where the subindices denote corestriction as in section 3.

Proposition 6.2.2. *α_X defines a natural transformation $\mathsf{res}_\varphi \Rightarrow \mathsf{res}_\psi$.*

Proof. By construction, α_X is a morphism of left C-comodules (a is a morphism of D-comodules by definition of representation, hence in particular of C-comodules).

The fact that the A-actions are preserved by α_X is the commutativity of the following diagram, which holds by

(1) conaturality of α

(2) associativity of a

$$\begin{tikzcd}
A\otimes^C X \arrow[r, "\cong"] \arrow[d, "\cong"'] & A\otimes^C D\otimes^D X \arrow[rr, "\mathsf{id}_A\otimes^C\alpha\otimes^D\mathsf{id}_X"] & & A\otimes^C B\otimes^D X \arrow[r, "\mathsf{id}_A\otimes^C a"] \arrow[d, "\cong"] & A\otimes^C X \arrow[d, "\cong"] \\
A\otimes^C D\otimes^D X \arrow[dd, "\varphi_1\otimes^D\mathsf{id}_X"'] & & & A\otimes^C D\otimes^D B\otimes^D X \arrow[d, "\psi_1\otimes^D\mathsf{id}_B\otimes^D\mathsf{id}_X"] & A\otimes^C D\otimes^D X \arrow[d, "\psi_1\otimes^D\mathsf{id}_X"] \\
 & (1) & & B\otimes^D B\otimes^D X \arrow[r, "\mathsf{id}_B\otimes^D a"] \arrow[d, "m\otimes^D\mathsf{id}_X"] & B\otimes^D X \arrow[dd, "a"] \\
B\otimes^D X \arrow[r, "\cong"] \arrow[d, "a"'] & D\otimes^D B\otimes^D X \arrow[r, "\alpha\otimes^D\mathsf{id}_B\otimes^D\mathsf{id}_X"] & B\otimes^D B\otimes^D X \arrow[r, "m\otimes^D\mathsf{id}_X"] \arrow[d, "\mathsf{id}_B\otimes^D a"] & B\otimes^D X \arrow[dr, "a"] \arrow[r, phantom, "(2)"] \arrow[d, phantom, "(2)"] & \\
X \arrow[r, "\cong"'] & D\otimes^D X \arrow[r, "\alpha\otimes^D\mathsf{id}_X"'] & B\otimes^D X \arrow[rr, "a"'] & & X
\end{tikzcd}.$$

Finally, naturality for α_X is the commutativity of the following diagram, where (1) commutes since, by assumption, $f : X \to Y$ is a morphism of $\mathfrak{D}$-representations:

$$\begin{tikzcd}
X \arrow[r, "\cong"] \arrow[d, "f"'] & D\otimes^D X \arrow[r, "\alpha\otimes^D\mathsf{id}_X"] \arrow[d, "\mathsf{id}_D\otimes^D f"] & B\otimes^D X \arrow[r, "a"] \arrow[d, "\mathsf{id}_B\otimes^D f"] \arrow[r, phantom, shift right=4ex, "(1)"] & X \arrow[d, "f"] \\
Y \arrow[r, "\cong"'] & D\otimes^D Y \arrow[r, "\alpha\otimes^D\mathsf{id}_Y"'] & B\otimes^D Y \arrow[r, "a"'] & Y
\end{tikzcd}.$$

□

In addition, one can similarly show that the construction of res preserves compositions: $\mathsf{res}_\varphi \circ \mathsf{res}_\psi = \mathsf{res}_{\psi\circ\varphi}$, for any pair of composable cofunctors φ and ψ. Also, composition of natural cotransformations corresponds to composition of the associated natural transformations. All these results can thus be summarized as follows.

Corollary 6.2.1. *Restriction is a 2-functor (contravariant on arrows and covariant on 2-cells)*

$$\overleftarrow{\mathit{Cat}}_{\mathbb{S}} \to \mathit{LCat}, \quad \mathfrak{C} \mapsto \mathsf{Rep}_{\mathbb{S}}\mathfrak{C}, \; \varphi \mapsto \mathsf{res}_{\varphi}, \; \alpha \mapsto \alpha \ .$$

Proof. Done above. ☐

Example 6.2.1. Recall from examples 6.1.1 that a representation of a category $\mathfrak{D} = (B, D, \dots)$ in Sets can be equivalently described as a D-graded set

$$X = \coprod_{d \in D} X_d \text{ equipped with maps } X_b : X_d \to X_{d'} \text{ for each arrow } b : d \to d' \text{ of } \mathfrak{D} \ ,$$

such that associativity and unitality are preserved in the obvious way. If $\varphi : \mathfrak{C} \to \mathfrak{D}$ is a cofunctor, then the $\mathfrak{C}$-representation $\mathsf{res}_{\varphi}(X)$ is the same set X, with C-grading defined by

$$X_c = \coprod_{d \in \varphi_0^{-1}(c)} X_d$$

and where the action of an arrow $a : c \to c'$ of $\mathfrak{C}$ on X_c is given by the action of $\varphi_1(a, d)$ on $X_d \subseteq X_c$ for each $d \in \varphi_0^{-1}(c)$. In pictures:

$$\begin{tikzcd}
\mathfrak{D} & d \arrow[r, bend left, "\varphi_1(a,d)"] \arrow[d, mapsto] & d' \arrow[d, mapsto] & \coprod X_d \arrow[r, "\coprod X_{\varphi_1(a,d)}"] \arrow[d, equal] & \coprod X_{d'} \arrow[d, equal] \\
\mathfrak{C} \arrow[u, "\varphi"] & c \arrow[r, bend left, "a"] & c' & X_c \arrow[r, dashed, "X_a"] & X_{c'}
\end{tikzcd} \ .$$

We close the section by briefly describing an analogous construction that can be carried out for functors instead of cofunctors. Namely, given a functor $f = (f_1, f_0) : \mathfrak{C} \to \mathfrak{D}$, one can again define a restriction functor

$$\mathsf{res}_f : \mathsf{Rep}_{\mathbb{S}}\mathfrak{D} \to \mathsf{Rep}_{\mathbb{S}}\mathfrak{C},$$

which now fits into a commutative diagram

$$\begin{tikzcd}
\mathsf{Rep}_{\mathbb{S}}\mathfrak{C} \arrow[r] & \mathsf{Comod}_{\mathbb{S}}C \\
\mathsf{Rep}_{\mathbb{S}}\mathfrak{D} \arrow[r] \arrow[u, "\mathsf{res}_f"] & \mathsf{Comod}_{\mathbb{S}}D \arrow[u, "\mathsf{coind}_{f_0}"']
\end{tikzcd} .$$

In particular res_f *does not* preserve the forgetful functors to $\mathbb{S}$, since coind_{f_0} does not, unless $f_0 = \mathsf{id}_C$, in which case f can be seen as a cofunctor and then the two types of restriction along f (as a functor or as a cofunctor) coincide.

Explicitly, given a $\mathfrak{D}$-representation X, its restriction along f is

$$\mathsf{res}_f(X) = C_f{\otimes^D}X$$

with its canonical structure of left C-comodule, and the following A-action:

$$A{\otimes^C}\mathsf{res}_f(X) = A{\otimes^C}C_f{\otimes^D}X \cong A_f{\otimes^D}X \xrightarrow{f_1\otimes^D\mathsf{id}_X} C_f{\otimes^D}B{\otimes^D}X \xrightarrow{\mathsf{id}_C\otimes^D a} C_f{\otimes^D}X = \mathsf{res}_f(X) \ .$$

Here, we are viewing the functor f as a pair (f_1, f_0) with $f_0 : C \to D$ and $f_1 : A_f \to C_f{\otimes^D}B$, as explained in section 4.3. Associativity of the A-action follows easily from that of the B-action plus functoriality of f, similarly for unitality.

One can proceed similarly with natural transformations. If $\alpha : f \Rightarrow g$ is a natural transformation between functors $\mathfrak{C} \to \mathfrak{D}$, so that (in the alternative notation of section 4.3) $\alpha : C_f \to C_g{\otimes^D}B$ is a morphism of C-D-bicomodules, one can define a natural transformation $\mathsf{res}_f \Rightarrow \mathsf{res}_g$ via

$$\alpha_X : C_f{\otimes^D}X \xrightarrow{\alpha\otimes^D\mathsf{id}_X} C_g{\otimes^D}B{\otimes^D}X \xrightarrow{\mathsf{id}_C\otimes^D a} C_g{\otimes^D}X \ .$$

Altogether this gives a 2-functor (contravariant on arrows, covariant on 2-cells)

$$\overrightarrow{\mathit{Cat}}_{\mathbb{S}} \to \mathit{LCat}\,, \quad \mathfrak{C} \mapsto \mathsf{Rep}_{\mathbb{S}}\mathfrak{C},\ f \mapsto \mathsf{res}_f,\ \alpha \mapsto \alpha \ .$$

6.3 Yoneda's lemma

The result below generalizes at the same time [Joh, proposition 2.21] and [P, theorem 2.2]. The former is the special case when $\mathcal{S}$ is lex, the latter is for arbitrary monoidal categories $\mathcal{S}$ but for monoids instead of internal categories (i.e. the special case $\mathfrak{C} = \underset{\vee}{A}$). For the definition of monads and monadic functors the reader is referred to [ML, chapter VI].

Proposition 6.3.1. *Let $\mathfrak{C}$ be a category in $\mathcal{S}$, with base comonoid C. Then the forgetful functor $\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \to \mathsf{Comod}_{\mathcal{S}}C$ is monadic.*

Proof. Let $T : \mathsf{Comod}_{\mathcal{S}}C \to \mathsf{Comod}_{\mathcal{S}}C$ be $T(X) = A \otimes^C X$. Define natural transformations $\mu : T^2 \Rightarrow T$ and $\eta : Id \Rightarrow T$ by

$$\mu_X : A \otimes^C A \otimes^C X \xrightarrow{m \otimes^C \mathsf{id}_X} A \otimes^C X \text{ and } \eta_X : X \xrightarrow{\cong} C \otimes^C X \xrightarrow{i \otimes^C \mathsf{id}_X} A \otimes^C X \ .$$

Then the axioms in the definition of internal category (2.3.1) immediately imply that (T, μ, η) is a monad in $\mathsf{Comod}_{\mathcal{S}}C$. Moreover, comparing the definition of algebras over a monad [ML, VI.2] with that of representations of an internal category (6.1.1), we see that there is an equivalence K as in the diagram below

$$\begin{tikzcd}
\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \arrow[dr] \arrow[dd, dashed, "K"'] & \\
& \mathsf{Comod}_{\mathcal{S}}C \\
(\mathsf{Comod}_{\mathcal{S}}C)^T \arrow[ur] &
\end{tikzcd} ,$$

where $(\mathsf{Comod}_{\mathcal{S}}C)^T$ is the category of T-algebras in $\mathsf{Comod}_{\mathcal{S}}C$. This means that $\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \to \mathsf{Comod}_{\mathcal{S}}C$ is monadic. □

We can now derive Yoneda's lemma for internal categories.

Corollary 6.3.1. *The forgetful functor* $\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \to \mathsf{Comod}_{\mathcal{S}}C$ *possesses a left adjoint* $\mathsf{Comod}_{\mathcal{S}}C \to \mathsf{Rep}_{\mathcal{S}}\mathfrak{C}$*, which sends* X *to* $A\otimes^C X$*, where* $A\otimes^C X$ *is viewed as* $\mathfrak{C}$*-representation via the following left* C*-comodule structure and* A*-action maps:*

$$A\otimes^C X \xrightarrow{s\otimes^C \mathsf{id}_X} C\otimes A\otimes^C X \ \text{ and } \ A\otimes^C A\otimes^C X \xrightarrow{m\otimes^C \mathsf{id}_X} A\otimes^C X \ .$$

Proof. This follows from proposition 6.3.1 plus [ML, theorem VI.2.1]. ☐

When $\mathcal{S} =$ ***Sets***, the above corollary specializes to the usual Yoneda lemma (for small categories). To explain this, assume that $\mathfrak{C}$ is a category in ***Sets***, let $F : \mathfrak{C} \to$ ***Sets*** be a functor and $u \in C$ an object of $\mathfrak{C}$. Recall (section 6.1) that F can equivalently be seen as a representation X_F of $\mathfrak{C}$. We can view u as a morphism of comonoids $u : I \to C$, and hence consider the corestricted C-comodule ${}_uI$. Then, the adjunction of corollary 6.3.1 implies in particular that

$$\mathsf{Hom}_{\mathfrak{C}}(A\otimes^C {}_uI, X_F) \cong \mathsf{Hom}_C({}_uI, X_F) \ .$$

Now, when viewed as a functor $\mathfrak{C} \to$ ***Sets***, the $\mathfrak{C}$-representation $A\otimes^C {}_uI$ is precisely the hom-functor $\mathfrak{C}(u, -)$, since

$$A\otimes^C {}_uI = \{a \in A \ / \ \tilde{s}(a) = u\} = \coprod_{v\in C}\{a \in A \ / \ \tilde{s}(a) = u, \ \tilde{t}(a) = v\} = \coprod_{v\in C} \mathfrak{C}(u, v) \ .$$

Also, by definition of X_F, $\mathsf{Hom}_C({}_uI, X_F) \cong F(u)$. Hence the bijection above becomes

$$\mathsf{Hom}(\mathfrak{C}(u, -), F) \cong F(u) \ ,$$

where Hom now denotes natural transformations. This is Yoneda's lemma [ML, lemma III.2].

The forgetful functor $\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \to \mathcal{S}$ does not always have a left adjoint. For instance, if $\mathcal{S} = \mathit{Vec}_k$ and $\mathfrak{C} = \widehat{\underset{\vee}{\mathrm{C}}}C$, then the result in corollary A.3.1 says that $\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \to \mathit{Vec}_k$ has a left adjoint if and only if C is finite-dimensional. Similarly, one can show that $\mathsf{Rep}_{\mathit{Sets}}\widehat{\underset{\vee}{\mathrm{C}}} \to \mathit{Sets}$ has a left adjoint if and only if $C = \{*\}$.

6.4 Representations and admissible sections

We expect representations of an internal category $\mathfrak{C}$ to be somehow related to representations of the monoid of admissible sections $\Gamma(\mathfrak{C})$. This cannot hold on the nose since $\mathsf{Rep}_{\mathcal{S}}\mathfrak{C}$ consists of objects of $\mathcal{S}$ while $\Gamma(\mathfrak{C})$ is an ordinary monoid (in Sets). In some cases it is possible to introduce an *internal* version of $\Gamma(\mathfrak{C})$ (a monoid in $\mathcal{S}$) for which there will be a functor

$$\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \to \mathsf{Mod}_{\mathcal{S}}(\Gamma(\mathfrak{C})) \ ;$$

for reasons of space this is not discussed in this work (unless for the case $\mathcal{S} = \mathit{Vec}_k$, which is addressed in section 9.2). In this section we discuss the "external" version of this; namely, for each representation X of $\mathfrak{C}$, we define a morphism of monoids

$$\Gamma(\mathfrak{C}) \to \mathsf{End}_{\mathcal{S}}(X) \ ,$$

which is natural with respect to cofunctors.

Proposition 6.4.1. *Let $\mathfrak{C} = (A, C, \dots)$ be a category in $\mathcal{S}$ and (X, p, a) a representation. Then there is a morphism of monoids $\gamma_X : \Gamma(\mathfrak{C}) \to \mathsf{End}_{\mathcal{S}}(X)$, where*

$$\gamma_X(u) : X \xrightarrow{p} C \otimes^C X \xrightarrow{u \otimes^C \mathsf{id}_X} A \otimes^C X \xrightarrow{a} X \ .$$

Proof. First, $\gamma_X(i) = \mathsf{id}_X$ precisely by unitality for the representation X. The fact that γ_X transforms multiplication of admissible sections u and v into composition of maps is the commutativity of the following diagram, which holds since

(1) a is a morphism of left C-comodules

(2) a is associative

$$\begin{tikzcd}
X \arrow[r, "p"] & C\otimes^C X \arrow[r, "v\otimes^C \mathsf{id}_X"] & A\otimes^C X \arrow[r, "t\otimes^C \mathsf{id}_X"] \arrow[d, "a"'] \arrow[dr, phantom, "(1)"] & C\otimes^C A\otimes^C X \arrow[r, "u\otimes^C \mathsf{id}_A\otimes^C \mathsf{id}_X"] \arrow[d, "\mathsf{id}_C\otimes^C a"] & A\otimes^C A\otimes^C X \arrow[r, "m\otimes^C \mathsf{id}_X"] \arrow[d, "\mathsf{id}_A\otimes^C a"] \arrow[dr, phantom, "(2)"] & A\otimes^C X \arrow[d, "a"] \\
 & & X \arrow[r, "p"'] & C\otimes^C X \arrow[r, "u\otimes^C \mathsf{id}_X"'] & A\otimes^C X \arrow[r, "a"'] & X
\end{tikzcd} .$$

$\square$

Next we show that γ is natural with respect to cofunctors.

Proposition 6.4.2. *Let* $\varphi : \mathfrak{C} \to \mathfrak{D}$ *be a cofunctor,* $Y \in \mathit{Rep}_{\mathbb{S}}\mathfrak{D}$ *and* $X = \mathit{res}_\varphi(Y) \in \mathit{Rep}_{\mathbb{S}}\mathfrak{C}$. *Then the following diagram commutes:*

$$\begin{tikzcd}
\Gamma(\mathfrak{C}) \arrow[r, "\gamma_X"] \arrow[d, "\Gamma(\varphi)"'] & \mathit{End}_{\mathbb{S}}(X) \arrow[d, equal] \\
\Gamma(\mathfrak{D}) \arrow[r, "\gamma_Y"'] & \mathit{End}_{\mathbb{S}}(Y)
\end{tikzcd} .$$

Proof. The desired result is the commutativity of the following diagram, where the top boundary is $\gamma_Y\Big(\Gamma(\varphi)(u)\Big)$ and the bottom $\gamma_X(u)$ ((1) and (2) commute by definition of res_φ):

$$\begin{tikzcd}
Y \arrow[r, "p_Y"] \arrow[d, equal] & D\otimes^D Y \arrow[r, "\Delta_D\otimes^D \mathsf{id}_Y"] & D\otimes^D D\otimes^D Y \arrow[r, "\varphi_0\otimes^D \mathsf{id}_D\otimes^D \mathsf{id}_Y"] \arrow[d, phantom, "(1)"] & C\otimes^C D\otimes^D Y \arrow[r, "u\otimes^C \mathsf{id}_D\otimes^D \mathsf{id}_Y"] \arrow[d, "\cong"] & A\otimes^C D\otimes^D Y \arrow[r, "\varphi_1\otimes^D \mathsf{id}_Y"] \arrow[d, "\cong"] \arrow[dr, phantom, "(2)"] & B\otimes^D Y \arrow[d, "a_Y"] \\
X \arrow[rrr, "p_X"'] & & & C\otimes^C X \arrow[r, "u\otimes^C \mathsf{id}_X"'] & A\otimes^C X \arrow[r, "a_X"'] & Y
\end{tikzcd} .$$

$\square$

Chapter 7

Tensor product of internal categories

In this chapter the regular monoidal category $\mathcal{S}$ is assumed to be symmetric, in addition. We will introduce a monoidal structure on $\overrightarrow{\mathcal{Cat}}_{\mathcal{S}}$ and $\overleftarrow{\mathcal{Cat}}_{\mathcal{S}}$ and study its relationship to admissible sections and representations. Deltacategories will be defined as comonoids in $\overleftarrow{\mathcal{Cat}}_{\mathcal{S}}$.

7.1 The monoidal 2-category of bicomodules

We proceed in an analogous way to that of chapter 2; namely, we first study the 2-category $\mathcal{G}$ of bicomodules and then specialize to graphs and categories. We will show that, when $\mathcal{S}$ is symmetric monoidal, $\mathcal{G}$ is a *monoidal 2-category*. For the definition of monoidal 2-categories in the general (non-strict) case see [KV]. In the strict case, a monoidal structure on $\mathcal{G}$ is a 2-functor $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, which is associative and unital in the obvious sense. In particular, the tensor product $\otimes$ is defined on objects, arrows and 2-cells, and preserves the vertical and horizontal composition and identities.

Recall (section 1.4) that, since $\mathcal{S}$ is symmetric monoidal, so is $\mathit{Comon}_{\mathcal{S}}$, under the same symmetry τ of $\mathcal{S}$. In particular, given comonoids $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$, the following defines a structure of comonoid on $C \otimes D$:

$$C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\mathsf{id}_C \otimes \tau_{C,D} \otimes \mathsf{id}_D} C \otimes D \otimes C \otimes D \text{ and } C \otimes D \xrightarrow{\epsilon_C \otimes \epsilon_D} I \otimes I = I \ .$$

Moreover, given a C_1-C_2-bicomodule (A, s_A, t_A) and a D_1-D_2-bicomodule (B, s_B, t_B), the following define a structure of $C_1 \otimes D_1$-$C_2 \otimes D_2$-bicomodule on $A \otimes B$:

$$A \otimes B \xrightarrow{t_A \otimes t_B} C_1 \otimes A \otimes D_1 \otimes B \xrightarrow{\mathsf{id}_{C_1} \otimes \tau_{A,D_1} \otimes \mathsf{id}_B} C_1 \otimes D_1 \otimes A \otimes B$$

and

$$A \otimes B \xrightarrow{s_A \otimes s_B} A \otimes C_2 \otimes B \otimes D_2 \xrightarrow{\mathsf{id}_A \otimes \tau_{C_2,B} \otimes \mathsf{id}_{D_2}} A \otimes B \otimes C_2 \otimes D_2 \ .$$

This defines the tensor product of $\mathcal{G}$ on objects and morphisms; schematically:

$$(C_2 \xrightarrow{A} C_1) \otimes (D_2 \xrightarrow{B} D_1) = C_2 \otimes D_2 \xrightarrow{A \otimes B} C_1 \otimes D_1 \ .$$

On 2-cells, we let $\otimes$ be just the tensor product of morphisms in $\mathcal{S}$. This clearly preserves vertical composition and identities, since these come from those of $\mathcal{S}$. Preservation of horizontal identities is also obvious:

$$(C \xrightarrow{C} C) \otimes (D \xrightarrow{D} D) = C \otimes D \xrightarrow{C \otimes D} C \otimes D \ .$$

On the other hand, preservation of the horizontal composition is the content of the following lemma.

Lemma 7.1.1. *In the situation*

$$C_2 \xrightarrow{A_2} C_1 \xrightarrow{A_1} C_0 \text{ and } D_2 \xrightarrow{B_2} D_1 \xrightarrow{B_1} D_0 \ ,$$

there is a canonical isomorphism of $C_0 \otimes D_0$-$C_2 \otimes D_2$-bicomodules

$$(A_1 \otimes^{C_1} A_2) \otimes (B_1 \otimes^{D_1} B_2) \cong (A_1 \otimes B_1) \otimes^{C_1 \otimes D_1} (A_2 \otimes B_2) \ .$$

Proof. We will show that the following composite

$$(A_1 \otimes^{C_1} A_2) \otimes (B_1 \otimes^{D_1} B_2) \xrightarrow{\mathsf{can} \otimes \mathsf{can}} A_1 \otimes A_2 \otimes B_1 \otimes B_2 \xrightarrow{\mathsf{id}_1 \otimes \tau \otimes \mathsf{id}_2} A_1 \otimes B_1 \otimes A_2 \otimes B_2$$

is the equalizer of the two rows below

$$\begin{tikzcd}
A_1 \otimes B_1 \otimes A_2 \otimes B_2 \arrow[r, "s_1 \otimes s_1 \otimes \mathsf{id}_{22}"] \arrow[dr, "\mathsf{id}_{11} \otimes t_2 \otimes t_2"'] & A_1 \otimes C_1 \otimes B_1 \otimes D_1 \otimes A_2 \otimes B_2 \arrow[r, "\mathsf{id}_1 \otimes \tau \otimes \mathsf{id}_D \otimes \mathsf{id}_{22}"] & A_1 \otimes B_1 \otimes C_1 \otimes D_1 \otimes A_2 \otimes B_2 \ , \\
 & A_1 \otimes B_1 \otimes C_1 \otimes A_2 \otimes D_1 \otimes B_2 \arrow[ur, "\mathsf{id}_{11} \otimes \mathsf{id}_C \otimes \tau \otimes \mathsf{id}_2"'] &
\end{tikzcd}$$

which by definition is $(A_1 \otimes B_1) \otimes^{C_1 \otimes D_1} (A_2 \otimes B_2)$. This will complete the proof.

To this end, consider the following diagram in $\mathcal{S}$

$$\begin{tikzcd}
(A_1 \otimes^{C_1} A_2) \otimes (B_1 \otimes^{D_1} B_2) \arrow[r, "\mathsf{can} \otimes \mathsf{id}_{12}"] \arrow[d, "\mathsf{id}_{12} \otimes \mathsf{can}"'] & (A_1 \otimes A_2) \otimes (B_1 \otimes^{D_1} B_2) \arrow[r, shift left, "s_1 \otimes \mathsf{id}_2 \otimes \mathsf{id}_{12}"] \arrow[r, shift right, "\mathsf{id}_1 \otimes t_2 \otimes \mathsf{id}_{12}"'] \arrow[d, "\mathsf{id}_{12} \otimes \mathsf{can}"'] & (A_1 \otimes C_1 \otimes A_2) \otimes (B_1 \otimes^{D_1} B_2) \arrow[d, "\mathsf{id}_{112} \otimes \mathsf{can}"] \\
(A_1 \otimes^{C_1} A_2) \otimes (B_1 \otimes B_2) \arrow[r, "\mathsf{can} \otimes \mathsf{id}_{12}"] \arrow[d, shift right, "\mathsf{id}_{12} \otimes \mathsf{id}_1 \otimes t_2"'] \arrow[d, shift left, "\mathsf{id}_{12} \otimes s_1 \otimes \mathsf{id}_2"] & (A_1 \otimes A_2) \otimes (B_1 \otimes B_2) \arrow[r, shift left, "s_1 \otimes \mathsf{id}_2 \otimes \mathsf{id}_{12}"] \arrow[r, shift right, "\mathsf{id}_1 \otimes t_2 \otimes \mathsf{id}_{12}"'] \arrow[d, shift right, "\mathsf{id}_{12} \otimes \mathsf{id}_1 \otimes t_2"'] \arrow[d, shift left, "\mathsf{id}_{12} \otimes s_1 \otimes \mathsf{id}_2"] & (A_1 \otimes C_1 \otimes A_2) \otimes (B_1 \otimes B_2) \arrow[d, shift right, "\mathsf{id}_{112} \otimes \mathsf{id}_1 \otimes t_2"'] \arrow[d, shift left, "\mathsf{id}_{112} \otimes s_1 \otimes \mathsf{id}_2"] \\
(A_1 \otimes^{C_1} A_2) \otimes (B_1 \otimes D_1 \otimes B_2) \arrow[r, "\mathsf{can} \otimes \mathsf{id}_{112}"'] & (A_1 \otimes A_2) \otimes (B_1 \otimes D_1 \otimes B_2) \arrow[r, shift left, "s_1 \otimes \mathsf{id}_2 \otimes \mathsf{id}_{112}"] \arrow[r, shift right, "\mathsf{id}_1 \otimes t_2 \otimes \mathsf{id}_{112}"'] & (A_1 \otimes C_1 \otimes A_2) \otimes (B_1 \otimes D_1 \otimes B_2)
\end{tikzcd}$$

Let us check the hypothesis in Johnstone's lemma 1.2.1. The squares commute as required, either by naturality of can or functoriality of $\otimes$. By regularity, all rows and columns of this diagram are equalizers. Also, $(\mathsf{id}_{112} \otimes s_1 \otimes \mathsf{id}_2, \mathsf{id}_{112} \otimes \mathsf{id}_1 \otimes t_2)$ and $(s_1 \otimes \mathsf{id}_2 \otimes \mathsf{id}_{112}, \mathsf{id}_1 \otimes t_2 \otimes \mathsf{id}_{112})$ are both coreflexive pairs, being split respectively by

$\mathsf{id}_{112}\otimes\mathsf{id}_1\otimes\epsilon_{D_1}\otimes\mathsf{id}_2$ and $\mathsf{id}_1\otimes\epsilon_{C_1}\otimes\mathsf{id}_2\otimes\mathsf{id}_{112}$. Thus Johnstone's lemma applies, and we deduce that

$$\begin{tikzcd}
(A_1\otimes^{C_1}A_2)\otimes(B_1\otimes^{D_1}B_2) \arrow[r, "\mathsf{can}\otimes\mathsf{can}"] & A_1\otimes A_2\otimes B_1\otimes B_2 \arrow[r, shift left, "s_1\otimes\mathsf{id}_2\otimes s_1\otimes\mathsf{id}_2"] \arrow[r, shift right, "\mathsf{id}_1\otimes t_2\otimes\mathsf{id}_1\otimes t_2"'] & A_1\otimes C_1\otimes A_2\otimes B_1\otimes D_1\otimes B_2
\end{tikzcd}$$

is an equalizer. This is the top boundary of the following diagram, which commutes since τ is a symmetry. It follows that the bottom boundary is also an equalizer, since the vertical arrows are isomorphisms. This is the desired conclusion.

$$\begin{tikzcd}
(A_1\otimes^{C_1}A_2)\otimes(B_1\otimes^{D_1}B_2) \arrow[r, "\mathsf{can}\otimes\mathsf{can}"] & A_1\otimes A_2\otimes B_1\otimes B_2 \arrow[rr, shift left, "s_1\otimes\mathsf{id}_2\otimes s_1\otimes\mathsf{id}_2"] \arrow[rr, shift right, "\mathsf{id}_1\otimes t_2\otimes\mathsf{id}_1\otimes t_2"'] \arrow[ddd, "\mathsf{id}_1\otimes\tau\otimes\mathsf{id}_2"'] & & A_1\otimes C_1\otimes A_2\otimes B_1\otimes D_1\otimes B_2 \arrow[d, "\mathsf{id}_1\otimes\mathsf{id}_C\otimes\tau\otimes\mathsf{id}_D\otimes\mathsf{id}_2"'] \\
& & & A_1\otimes C_1\otimes B_1\otimes A_2\otimes D_1\otimes B_2 \arrow[dd, "\mathsf{id}_1\otimes\tau\otimes\tau\otimes\mathsf{id}_2"'] \\
& & A_1\otimes C_1\otimes B_1\otimes D_1\otimes A_2\otimes B_2 \arrow[dr, "\mathsf{id}_1\otimes\tau\otimes\mathsf{id}_D\otimes\mathsf{id}_{22}"] & \\
& A_1\otimes B_1\otimes A_2\otimes B_2 \arrow[ur, "s_1\otimes s_1\otimes\mathsf{id}_{22}"] \arrow[dr, "\mathsf{id}_{11}\otimes t_2\otimes t_2"'] & & A_1\otimes B_1\otimes C_1\otimes D_1\otimes A_2\otimes B_2 \\
& & A_1\otimes B_1\otimes C_1\otimes A_2\otimes D_1\otimes B_2 \arrow[ur, "\mathsf{id}_{11}\otimes\mathsf{id}_C\otimes\tau\otimes\mathsf{id}_2"'] &
\end{tikzcd}$$

□

The isomorphism of lemma 7.1.1 will be denoted by

$$\tau_{(A_1,A_2,B_1,B_2)} : (A_1\otimes^{C_1}A_2)\otimes(B_1\otimes^{D_1}B_2) \xrightarrow{\cong} (A_1\otimes B_1)\otimes^{C_1\otimes D_1}(A_2\otimes B_2)\ .$$

It is natural in all A_1, A_2, B_1 and B_2. Since it is induced by $\mathsf{id}_{A_1}\otimes\tau_{A_2,B_1}\otimes\mathsf{id}_{B_2}$, it is valid to use the same notation $\tau_{(A_1,B_1,A_2,B_2)}$ for its inverse, which is induced by $\mathsf{id}_{A_1}\otimes\tau_{B_1,A_2}\otimes\mathsf{id}_{B_2}$.

As explained above, this completes the proof of the fact that the 2-category $\mathcal{G}$ of bicomodules is monoidal under the tensor product described above. In particular, for any comonoids C and D in $\mathcal{S}$, there is a functor

$$\mathcal{G}_C \times \mathcal{G}_D \xrightarrow{\otimes} \mathcal{G}_{C\otimes D},\ (A,B)\mapsto A\otimes B,$$

where $\mathcal{G}_C = \mathcal{G}(C, C)$ is the category of internal graphs over C. Recall that $\mathcal{G}_C$ is in turn a monoidal category, under $\otimes^C$. Thus, the fact that $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ preserves horizontal composition implies that $\otimes : \mathcal{G}_C \times \mathcal{G}_D \to \mathcal{G}_{C \otimes D}$ is a monoidal functor. Since monoids are preserved under monoidal functors, it follows that $\otimes$ carries internal categories to internal categories. Explicitly, if $\mathfrak{C} = (A, C, \dots)$ and $\mathfrak{D} = (B, D, \dots)$ are categories in $\mathcal{S}$ then their tensor product $\mathfrak{C} \otimes \mathfrak{D} = (A \otimes B, C \otimes D, s, t, i, m)$ is as follows:

- $C \otimes D$ is a comonoid via

  $$C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\mathsf{id}_C \otimes \tau_{C,D} \otimes \mathsf{id}_D} C \otimes D \otimes C \otimes D \text{ and } C \otimes D \xrightarrow{\epsilon_C \otimes \epsilon_D} I \otimes I = I \ ,$$

- $t : A \otimes B \xrightarrow{t_A \otimes t_B} C \otimes A \otimes D \otimes B \xrightarrow{\mathsf{id}_C \otimes \tau_{A,D} \otimes \mathsf{id}_B} C \otimes D \otimes A \otimes B$ and

  $s : A \otimes B \xrightarrow{s_A \otimes s_B} A \otimes C \otimes B \otimes D \xrightarrow{\mathsf{id}_A \otimes \tau_{C,B} \otimes \mathsf{id}_D} A \otimes B \otimes C \otimes D$,

- $i : C \otimes D \xrightarrow{i_A \otimes i_B} A \otimes B$, and

- $m : (A \otimes B) \otimes^{C \otimes D} (A \otimes B) \xrightarrow{\tau_{(A,B,A,B)}} (A \otimes^C A) \otimes (B \otimes^D B) \xrightarrow{m_A \otimes m_B} A \otimes B$.

Moreover, it is clear that if $f = (f_1, f_0) : \mathfrak{C} \to \mathfrak{C}'$ and $g = (g_1, g_0) : \mathfrak{D} \to \mathfrak{D}'$ are functors, then so is

$$f \otimes g = (f_1 \otimes g_1, f_0 \otimes g_0) : \mathfrak{C} \otimes \mathfrak{D} \to \mathfrak{C}' \otimes \mathfrak{D}' \ .$$

Thus $\overrightarrow{\mathcal{C}at}_{\mathcal{S}}$ is a monoidal category; the unit object being $\mathfrak{I} = \hat{\mathrm{I}} = \underset{\vee}{\mathrm{I}}$.

Similarly, if $\varphi = (\varphi_1, \varphi_0) : \mathfrak{C} \to \mathfrak{C}'$ and $\psi = (\psi_1, \psi_0) : \mathfrak{D} \to \mathfrak{D}'$ are cofunctors, then so is $\varphi \otimes \psi : \mathfrak{C} \otimes \mathfrak{D} \to \mathfrak{C}' \otimes \mathfrak{D}'$, where $(\varphi \otimes \psi)_0 : D \otimes D' \xrightarrow{\varphi_0 \otimes \psi_0} C \otimes C'$ and

$$(\varphi \otimes \psi)_1 : (A \otimes B) \otimes^{C \otimes D} (C' \otimes D') \xrightarrow{\tau_{(A,B,C',D')}} (A \otimes^C C') \otimes (B \otimes^D D') \xrightarrow{\varphi_1 \otimes \psi_1} A' \otimes B'.$$

Thus $\overleftarrow{\mathcal{C}at}_{\mathcal{S}}$ is a monoidal category; the unit object being again $\mathfrak{I}$.

Finally, notice that there is a functor

$$\tau_{\mathfrak{C},\mathfrak{D}} : \mathfrak{C}\otimes\mathfrak{D} \to \mathfrak{D}\otimes\mathfrak{C} \text{ given by } (\tau_{A,B}, \tau_{C,D}) \ .$$

This is an isomorphism; in fact, $\tau_{\mathfrak{D},\mathfrak{C}} \circ \tau_{\mathfrak{C},\mathfrak{D}} = \mathsf{id}_{\mathfrak{C}\otimes\mathfrak{D}}$. Thus $\mathfrak{C}\otimes\mathfrak{D} \cong \mathfrak{D}\otimes\mathfrak{C}$ in $\overrightarrow{\mathcal{C}at}_{\mathcal{S}}$, and hence also in $\overleftarrow{\mathcal{C}at}_{\mathcal{S}}$ (isomorphisms in these categories coincide, by remark 4.2.1). This turns $\overrightarrow{\mathcal{C}at}_{\mathcal{S}}$ and $\overleftarrow{\mathcal{C}at}_{\mathcal{S}}$ into symmetric monoidal categories.

We should also remark that when $\mathcal{S} = \mathcal{S}ets$, the tensor product of categories just described boils down to the usual product of categories as in [ML, II.3].

7.2 Products and admissible sections

We begin by discussing one particular example of tensor product of categories, that shows that convolution of maps is a particular case of multiplication of admissible sections. If C is a comonoid in $\mathcal{S}$ with comultiplication Δ_C and counit ϵ_C, then C^{cop} denotes the same object C but viewed as a comonoid with comultiplication $\Delta_{C^{cop}} : C \xrightarrow{\Delta_C} C\otimes C \xrightarrow{\tau_{C,C}} C\otimes C$ and the same counit $\epsilon_{C^{cop}} = \epsilon_C$.

Example 7.2.1. Let A be a monoid and C a comonoid in $\mathcal{S}$. Then there is an isomorphism of monoids

$$\Gamma(\underset{\vee}{\mathrm{A}}\otimes\widehat{\mathrm{C}}) \cong \mathsf{Hom}_{\mathcal{S}}(C^{cop}, A)$$

where $\mathsf{Hom}_{\mathcal{S}}(C^{cop}, A)$ is a monoid under convolution.

Proof. By remark 3.0.1, there is a bijection

$$\Gamma(\underset{\vee}{\mathrm{A}}\otimes\widehat{\mathrm{C}}) = \mathsf{Hom}_C(C, A\otimes C) = \mathsf{Hom}_{\mathcal{S}}(C, A),\ u \mapsto \tilde{u} := (\mathsf{id}_A\otimes\epsilon_C)\circ u\ .$$

We only need to check that this is a morphism of monoids. The unit of $\Gamma(\underset{\vee}{\mathrm{A}}\otimes\widehat{\mathrm{C}})$ is $u_A\otimes\mathsf{id}_C$, so it gets sent to $(\mathsf{id}_A\otimes\epsilon_C)\circ(u_A\otimes\mathsf{id}_C) = u_A\circ\epsilon_C$, which is the unit of $\mathsf{Hom}_{\mathcal{S}}(C^{cop}, A)$. The fact that multiplication of admissible sections corresponds to convolution in $\mathsf{Hom}(C^{cop}, A)$ is the commutativity of the following diagram, where the top boundary is $\widetilde{u * v}$ and the bottom $\tilde{u} * \tilde{v}$:

$$\begin{tikzcd}
C \arrow[r, "v"] \arrow[d, "\Delta_C"] & A\otimes C \arrow[r, "\mathsf{id}_A\otimes\Delta_C"] \arrow[d, "\mathsf{id}_A\otimes\Delta_C"] & A\otimes(C\otimes^C C) \arrow[r, "\tau_{(I,A,C,C)}"] \arrow[d, "\mathsf{id}_A\otimes\mathsf{id}_C\otimes\epsilon_C"] & C\otimes^C(A\otimes C) \arrow[r, "u\otimes^C\mathsf{id}_{A\otimes C}"] \arrow[d, "\mathsf{id}_C\otimes\mathsf{id}_A\otimes\epsilon_C"] & (A\otimes C)\otimes^C(A\otimes C) \arrow[r, "\tau_{(A,C,A,C)}"] \arrow[d, "\mathsf{id}_A\otimes\mathsf{id}_C\otimes\mathsf{id}_A\otimes\epsilon_C"] & (A\otimes A)\otimes(C\otimes^C C) \arrow[r, "\mu_A\otimes\Delta_C^{-1}"] \arrow[d, "\mathsf{id}_A\otimes\mathsf{id}_A\otimes\epsilon_C\otimes\epsilon_C"] & A\otimes C \arrow[d, "\mathsf{id}_A\otimes\epsilon_C"] \\
C\otimes C \arrow[r, "v\otimes\mathsf{id}_C"'] \arrow[d, "\tau_{C,C}"] & A\otimes C\otimes C \arrow[r, "\mathsf{id}_A\otimes\epsilon_C\otimes\mathsf{id}_C"'] \arrow[d, "\tau_{A\otimes C,C}"] & A\otimes C \arrow[r, "\tau_{A,C}"'] & C\otimes A \arrow[r, "u\otimes\mathsf{id}_A"'] & A\otimes C\otimes A \arrow[r, "\mathsf{id}_A\otimes\epsilon_C\otimes\mathsf{id}_A"'] & A\otimes A \arrow[r, "\mu_A"'] & A \\
C\otimes C \arrow[r, "\mathsf{id}_C\otimes v"'] & C\otimes A\otimes C \arrow[urr, bend right, "\mathsf{id}_C\otimes\mathsf{id}_A\otimes\epsilon_C"'] & & & & &
\end{tikzcd}$$

(The first square commutes by definition of admissible section: v is a morphism of right C-comodules; the others commute by functoriality of $\otimes$, naturality of τ, or counitality of C). ☐

We now study the behavior of admissible sections with respect to products. Let $\mathfrak{C}$ and $\mathfrak{D}$ be two categories in $\mathcal{S}$. The monoidal structure on $\mathcal{G}$ (section 7.1) yields in particular a map

$$\Gamma(\mathfrak{C})\times\Gamma(\mathfrak{D}) = \mathsf{Hom}_C(C, A)\times\mathsf{Hom}_D(D, B) \xrightarrow{\otimes} \mathsf{Hom}_{C\otimes D}(C\otimes D, A\otimes B) = \Gamma(\mathfrak{C}\otimes\mathfrak{D})\ .$$

Proposition 7.2.1. *The map* $\Gamma(\mathfrak{C})\times\Gamma(\mathfrak{D}) \xrightarrow{\otimes} \Gamma(\mathfrak{C}\otimes\mathfrak{D})$ *is a morphism of monoids. Moreover, it is natural with respect to cofunctors, in the sense that if* $\varphi : \mathfrak{C} \to \mathfrak{C}'$

and $\psi : \mathfrak{D} \to \mathfrak{D}'$ *are cofunctors, then the following diagram commutes:*

$$\begin{tikzcd}
\Gamma(\mathfrak{C}) \times \Gamma(\mathfrak{D}) \arrow[r, "\otimes"] \arrow[d, "\Gamma(\varphi)\times\Gamma(\psi)"'] & \Gamma(\mathfrak{C}\otimes\mathfrak{D}) \arrow[d, "\Gamma(\varphi\otimes\psi)"] \\
\Gamma(\mathfrak{C}') \times \Gamma(\mathfrak{D}') \arrow[r, "\otimes"'] & \Gamma(\mathfrak{C}'\otimes\mathfrak{D}')
\end{tikzcd}$$

Proof. The fact that $\otimes$ preserves unit elements is direct from the definition of identities for $\mathfrak{C}\otimes\mathfrak{D}$. Multiplications are preserved too, since in the following commutative diagram, the top boundary is $(u * u')\otimes(v * v')$ and the bottom $(u\otimes v) * (u'\otimes v')$, where $u, u' \in \Gamma(\mathfrak{C})$ and $v, v' \in \Gamma(\mathfrak{D})$ are arbitrary:

$$\begin{tikzcd}
C\otimes D \arrow[r, "u'\otimes v'"] & A\otimes B \arrow[r, "t_A\otimes t_B"] \arrow[dr, "t_{A\otimes B}"'] & (C\otimes^C A)\otimes(D\otimes^D B) \arrow[r, "u\otimes^C \mathsf{id}_A\otimes v\otimes^D \mathsf{id}_B"] \arrow[d, "{\tau_{(C,A,D,B)}}"] & (A\otimes^C A)\otimes(B\otimes^D B) \arrow[r, "m_A\otimes m_B"] \arrow[d, "{\tau_{(A,A,B,B)}\otimes \mathsf{id}_B}"] & A\otimes B \; . \\
 & & (C\otimes D)\otimes^{C\otimes D}(A\otimes B) \arrow[r, "(u\otimes v)\otimes^{C\otimes D}\mathsf{id}_{A\otimes B}"'] & (A\otimes B)\otimes^{C\otimes D}(A\otimes B) \arrow[ur, "m_{A\otimes B}"'] &
\end{tikzcd}$$

Finally, the commutativity of the diagram below yields the desired naturality with respect to cofunctors, since the top is $\Gamma(\varphi\otimes\psi)(u\otimes v)$ and the bottom $\Gamma(\varphi)(u)\otimes\Gamma(\psi)(v)$.

$$\begin{tikzcd}
C'\otimes D' \cong (C\otimes D)\otimes^{C\otimes D}(C'\otimes D') \arrow[r, "u\otimes v"] \arrow[dr, "{\tau_{(()}C,D,C',D')}"'] & (A\otimes B)\otimes^{C\otimes D}(C'\otimes D') \arrow[r, "{\tau_{(()}A,B,C',D')}"] & (A\otimes^C C')\otimes(B\otimes^D D') \arrow[r, "\varphi_1\otimes\psi_1"] & A'\otimes B' \\
 & (C\otimes^C C')\otimes(D\otimes^D D') \arrow[ur, "u\otimes^C \mathsf{id}_{C'}\otimes v\otimes^D \mathsf{id}_{D'}"'] & &
\end{tikzcd}$$

$\square$

The map $\Gamma(\mathfrak{C}) \times \Gamma(\mathfrak{D}) \to \Gamma(\mathfrak{C}\otimes\mathfrak{D})$ is not an isomorphism in general. For instance, consider the case $\mathcal{S} =$ ***Sets***, $\mathfrak{C} = \underset{\smile}{\mathrm{G}}$ and $\mathfrak{D} = \widehat{\mathrm{X}}$, where G is a monoid and X a set. Then we know from section 5.2 and example 7.2.1 that

$$\Gamma(\mathfrak{C}) = G, \quad \Gamma(\mathfrak{D}) = \{*\} \quad \text{while} \quad \Gamma(\mathfrak{C} \times \mathfrak{D}) = G^X,$$

the set of all maps $X \to G$ under point-wise multiplication (since that is what convolution boils down to in this case). Hence $\Gamma(\mathfrak{C}) \times \Gamma(\mathfrak{D})$ is far from $\Gamma(\mathfrak{C} \times \mathfrak{D})$. The lack of duals in Sets is responsible for this behavior. On the other hand, let us look at the "same" example when $\mathcal{S} = \mathit{Vec}_k$: $\mathfrak{C} = \underset{\vee}{\mathrm{A}}$, $\mathfrak{D} = \widehat{\mathrm{C}}$, where A is a k-algebra and C a k-coalgebra. Then

$$\Gamma(\mathfrak{C}) = A, \quad \Gamma(\mathfrak{D}) = (C^*)^{op} \quad \text{and} \quad \Gamma(\mathfrak{C} \otimes \mathfrak{D}) = \mathsf{Hom}_k(C^{cop}, A),$$

from where we see that $\Gamma(\mathfrak{C}) \times \Gamma(\mathfrak{D}) \to \Gamma(\mathfrak{C} \otimes \mathfrak{D})$ induces an isomorphism

$$\Gamma(\mathfrak{C}) \otimes_k \Gamma(\mathfrak{D}) \to \Gamma(\mathfrak{C} \otimes \mathfrak{D})$$

if and only if C is finite-dimensional. In fact, this turns out to be true for arbitrary categories in Vec_k as long as the base coalgebras C and D are finite-dimensional, as we will see in section 9.2. (Notice that in this case $\Gamma(\mathfrak{C})$ is not just a monoid but a k-algebra, under addition of linear maps).

7.3 Products and representations

Proposition 7.3.1. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories in $\mathcal{S}$. Consider two representations X of $\mathfrak{C}$ and Y of $\mathfrak{D}$. Then $X \otimes Y$ becomes a $\mathfrak{C} \otimes \mathfrak{D}$-representation when equipped with the following $C \otimes D$-comodule structure and $A \otimes B$-action:*

$$p_{X \otimes Y} : X \otimes Y \xrightarrow{p_X \otimes p_Y} C \otimes X \otimes D \otimes Y \cong C \otimes D \otimes X \otimes Y$$

and

$$a_{X\otimes Y} : (A\otimes B)\otimes^{C\otimes D}(X\otimes Y) \cong (A\otimes^{C}X)\otimes(B\otimes^{D}Y) \xrightarrow{a_X\otimes a_Y} X\otimes Y \ .$$

Proof. We have to check the axioms in definition 6.1.1. The considerations of section 7.1 (on the monoidal structure of $\mathcal{G}$) show that $X\otimes Y$ is a left $C\otimes D$-comodule with structure map $p_{X\otimes Y}$ and also that $a_{X\otimes Y}$ is a morphism of $C\otimes D$-comodules (this uses lemma 7.1.1). Unitaliy and associativity for $a_{X\otimes Y}$ boil down to those of a_X and a_Y, plus some obvious naturality properties of the isomorphism of lemma 7.1.1 (which in turn follow from those of the symmetry τ). ☐

Thus, for any two categories $\mathfrak{C}$ and $\mathfrak{D}$ there is a functor

$$\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \times \mathsf{Rep}_{\mathcal{S}}\mathfrak{D} \overset{\otimes}{\longrightarrow} \mathsf{Rep}_{\mathcal{S}}(\mathfrak{C}\otimes\mathfrak{D}) \ .$$

This may be seen as a natural transformation between the (large) contravariant functors $\mathsf{Rep}_{\mathcal{S}}(-) \times \mathsf{Rep}_{\mathcal{S}}(-)$ and $\mathsf{Rep}_{\mathcal{S}}(-\otimes-) : \boldsymbol{Cat}_{\mathcal{S}} \times \boldsymbol{Cat}_{\mathcal{S}} \to \boldsymbol{LCat}$, where $\boldsymbol{Cat}_{\mathcal{S}}$ denotes either $\overrightarrow{\boldsymbol{Cat}}_{\mathcal{S}}$ or $\overleftarrow{\boldsymbol{Cat}}_{\mathcal{S}}$, in view of the fact that the following diagrams commute (where $\varphi : \mathfrak{C} \to \mathfrak{C}'$ and $\psi : \mathfrak{D} \to \mathfrak{D}'$ are either functors or cofunctors):

$$\begin{tikzcd}
\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \times \mathsf{Rep}_{\mathcal{S}}\mathfrak{D} \arrow[r, "\otimes"] & \mathsf{Rep}_{\mathcal{S}}(\mathfrak{C}\otimes\mathfrak{D}) \\
\mathsf{Rep}_{\mathcal{S}}(\mathfrak{C}') \times \mathsf{Rep}_{\mathcal{S}}(\mathfrak{D}') \arrow[r, "\otimes"'] \arrow[u, "\mathsf{res}_{\varphi}\times\mathsf{res}_{\psi}"] & \mathsf{Rep}_{\mathcal{S}}(\mathfrak{C}'\otimes\mathfrak{D}') \arrow[u, "\mathsf{res}_{\varphi\otimes\psi}"']
\end{tikzcd} \ .$$

These assertions follow readily from lemma 7.1.1, complemented with routine manipulations.

We close the section by relating products of representations and admissible sections of products, via the canonical map γ of proposition 6.4.1.

Proposition 7.3.2. *Let X and Y be representations of two categories $\mathfrak{C}$ and $\mathfrak{D}$ and view $X \otimes Y$ as a $\mathfrak{C} \otimes \mathfrak{D}$-representation as above. Then the following diagram commutes:*

$$\begin{tikzcd}
\Gamma(\mathfrak{C}) \times \Gamma(\mathfrak{D}) \arrow[r, "\gamma_X \times \gamma_Y"] \arrow[d, "\otimes"'] & \mathit{End}_{\mathcal{S}}(X) \times \mathit{End}_{\mathcal{S}}(Y) \arrow[d, "\otimes"] \\
\Gamma(\mathfrak{C} \otimes \mathfrak{D}) \arrow[r, "\gamma_{X \otimes Y}"'] & \mathit{End}_{\mathcal{S}}(X \otimes Y)
\end{tikzcd}$$

Proof. In the following diagram, the top is $\gamma_X(u) \otimes \gamma_Y(v)$ and the bottom $\gamma_{X \otimes Y}(u \otimes v)$. The diagram commutes by naturality of the isomorphism in lemma 7.1.1 and definition of $p_{X \otimes Y}$ and $a_{X \otimes Y}$.

$$\begin{tikzcd}
X \otimes Y \arrow[r, "p_X \otimes p_Y"] \arrow[dr, "p_{X \otimes Y}"'] & (C \otimes^C X) \otimes (D \otimes^D Y) \arrow[r, "u \otimes^C \mathsf{id}_X \otimes v \otimes^D \mathsf{id}_Y"] \arrow[d, "\cong"] & (A \otimes^C X) \otimes (B \otimes^D Y) \arrow[r, "a_X \otimes a_Y"] \arrow[d, "\cong"] & X \otimes Y \; . \\
 & (C \otimes D) \otimes^{C \otimes D} (X \otimes Y) \arrow[r, "(u \otimes v) \otimes^{C \otimes D} \mathsf{id}_{X \otimes Y}"'] & (A \otimes B) \otimes^{C \otimes D} (X \otimes Y) \arrow[ur, "a_{X \otimes Y}"'] &
\end{tikzcd}$$

□

7.4 Deltacategories

Let $\mathcal{S}$ continue to be a symmetric regular monoidal category. In section 7.1 it was stablished that then $\overleftarrow{\mathit{Cat}}_{\mathcal{S}}$ is a monoidal category.

Definition 7.4.1. A deltacategory in $\mathcal{S}$ is a comonoid in $\overleftarrow{\mathit{Cat}}_{\mathcal{S}}$.

Notice that by construction of the monoidal structure of $\overleftarrow{\mathit{Cat}}_{\mathcal{S}}$, the forgetful functor $(\overleftarrow{\mathit{Cat}}_{\mathcal{S}})^{op} \to \mathit{Comon}_{\mathcal{S}}$, $\mathfrak{C} = (A, C, \dots) \mapsto C$, is monoidal. In particular, if $(\mathfrak{C}, \Delta, \epsilon)$ is a deltacategory with coassociative and counital cofunctors $\Delta = (\Delta_1, \Delta_0) : \mathfrak{C} \to \mathfrak{C} \otimes \mathfrak{C}$ and $\epsilon = (\epsilon_1, \epsilon_0) : \mathfrak{C} \to \mathfrak{I}$, then the comonoid C is actually a bimonoid, with multiplication $\Delta_0 : C \otimes C \to C$ and unit $\epsilon_0 : I \to C$. In addition, $\Delta_1 : A \otimes^C {}_{\Delta}(C \otimes C) \to {}_{\Delta}(A \otimes A)$

is a morphism of C-$C{\otimes}C$-bicomodules (where the subindex denotes corestriction along Δ_0), $\epsilon_1 : A{\otimes}^C{}_{\epsilon}I \to {}_{\epsilon}I$ is a morphism of left C-comodules (where the subindex denotes corestriction along ϵ_0), and these are such that seven diagrams commute, expressing the facts that Δ and ϵ are coassociative and counital cofunctors. Instead of making these conditions explicit, which is not very illuminating, we will present several different examples of deltacategories in $\mathtt{Sets}$ in chapter 8. These will be complemented with important examples of deltacategories in $\mathtt{Vec}_k$ in chapter 9.

Example 7.4.1. Bimonoids in $\mathcal{S}$ provide trivial examples of deltacategories in $\mathcal{S}$. In fact, it is clear that the fully-faithful functors

$$(\mathtt{Comon}_{\mathcal{S}})^{op} \to \overleftarrow{\mathtt{Cat}}_{\mathcal{S}},\ C \mapsto \widehat{\mathrm{C}}, \qquad \mathtt{Mon}_{\mathcal{S}} \to \overleftarrow{\mathtt{Cat}}_{\mathcal{S}},\ A \mapsto \underset{\vee}{\mathrm{A}}$$

of example 4.2.1 are monoidal; hence,

$$H \text{ is a bimonoid } \Leftrightarrow \widehat{\mathrm{H}} \text{ is a deltacategory } \Leftrightarrow \underset{\vee}{\mathrm{H}} \text{ is a deltacategory.}$$

In section 7.3 we constructed a functor

$$\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \times \mathsf{Rep}_{\mathcal{S}}\mathfrak{D} \xrightarrow{\otimes} \mathsf{Rep}_{\mathcal{S}}(\mathfrak{C}{\otimes}\mathfrak{D}),$$

which yields a natural transformation between the functors $\mathsf{Rep}_{\mathcal{S}}(-) \times \mathsf{Rep}_{\mathcal{S}}(-)$ and $\mathsf{Rep}_{\mathcal{S}}(-{\otimes}-) : \overleftarrow{\mathtt{Cat}}^{op}_{\mathcal{S}} \times \overleftarrow{\mathtt{Cat}}^{op}_{\mathcal{S}} \to \mathtt{LCat}$. This turns the functor

$$\mathsf{Rep}_{\mathcal{S}}(-) : \overleftarrow{\mathtt{Cat}}^{op}_{\mathcal{S}} \to \mathtt{LCat}$$

into a *lax monoidal functor*, in the sense that for categories $\mathfrak{C}$, $\mathfrak{D}$ and $\mathfrak{E}$, the following diagram (clearly) commutes:

$$\begin{tikzcd}
\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \times \mathsf{Rep}_{\mathcal{S}}\mathfrak{D} \times \mathsf{Rep}_{\mathcal{S}}\mathfrak{E} \arrow[r, "\otimes\times\mathsf{id}"] \arrow[d, "\mathsf{id}\times\otimes"'] & \mathsf{Rep}_{\mathcal{S}}(\mathfrak{C}\otimes\mathfrak{D}) \times \mathsf{Rep}_{\mathcal{S}}\mathfrak{E} \arrow[d, "\otimes"] \\
\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \times \mathsf{Rep}_{\mathcal{S}}(\mathfrak{D}\otimes\mathfrak{E}) \arrow[r, "\otimes"'] & \mathsf{Rep}_{\mathcal{S}}(\mathfrak{C}\otimes\mathfrak{D}\otimes\mathfrak{E})\ .
\end{tikzcd}$$

It is well-known that lax monoidal functors preserve monoids (but not comonoids, unless one requires the natural transformation in question to be invertible). We thus obtain that:

Corollary 7.4.1. *If* $\mathfrak{C}$ *is a deltacategory, then* $\mathsf{Rep}_{\mathcal{S}}\mathfrak{C}$ *is a monoidal category, in such a way that the forgetful functor* $\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \to \mathcal{S}$ *is monoidal.*

Proof. According to the discussion above, if $\Delta : \mathfrak{C} \to \mathfrak{C}\otimes\mathfrak{C}$ and $\epsilon : \mathfrak{C} \to \mathfrak{I}$ are the structure cofunctors of $\mathfrak{C}$, then $\mathsf{Rep}_{\mathcal{S}}\mathfrak{C}$ is monoidal under the tensor product

$$\mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \times \mathsf{Rep}_{\mathcal{S}}\mathfrak{C} \xrightarrow{\otimes} \mathsf{Rep}_{\mathcal{S}}(\mathfrak{C}\otimes\mathfrak{C}) \xrightarrow{\mathsf{res}_{\Delta}} \mathsf{Rep}_{\mathcal{S}}\mathfrak{C}\ ,$$

with unit object $I = \mathsf{res}_{\epsilon}(I)$. The forgetful functor is monoidal because res_{Δ} and res_{ϵ} preserve the forgetful functors (a general fact for cofunctors, section 6.2). ☐

The situation for admissible sections of deltacategories is more complicated. Consider the case $\mathcal{S} = \mathtt{Vec}_k$. The natural transformation $\Gamma(\mathfrak{C}) \times \Gamma(\mathfrak{D}) \xrightarrow{\otimes} \Gamma(\mathfrak{C}\otimes\mathfrak{D})$ of section 7.2 induces another natural transformation

$$\Gamma(\mathfrak{C})\otimes_k\Gamma(\mathfrak{D}) \xrightarrow{\otimes} \Gamma(\mathfrak{C}\otimes\mathfrak{D})$$

(recall that in this case one can view admissible sections as a functor $\Gamma : \overleftarrow{\mathtt{Cat}}_{\mathtt{Vec}_k} \to \mathtt{Alg}_k$). This turns the functor $\Gamma : \overleftarrow{\mathtt{Cat}}_{\mathcal{S}} \to \mathtt{Alg}_k$ into a lax monoidal functor. As pointed out above, this can be used to equip $\Gamma(\mathfrak{C})$ with a structure of k-bialgebra only if $\Gamma(\mathfrak{C})_{\otimes_k}\Gamma(\mathfrak{C}) \to \Gamma(\mathfrak{C}_{\otimes}\mathfrak{C})$ is an isomorphism. As already announced, this is the case when C is finite-dimensional. This is how many interesting quantum groups arise. We will elaborate on this in chapter 9.

Chapter 8

Deltacategories in *Sets*

Deltacategories in *Sets* are interesting objects deserving further study. In this chapter we limit ourselves to presenting a few families of examples.

We summarize the distinctive properties of deltacategories in *Sets* in table 8.1. This shows the relative position among the set-theoretic notions of monoids, deltacategories and categories.

Here, linearization means passage to a linear category (a category in Vec_k) and then to admissible sections. For the case of monoids this simply yields the usual monoid-algebra. Unlike the case of arbitrary small categories, the category of representations of a monoid or deltacategory is monoidal in such a way that the forgetful functor to *Sets* (or to Vec_k, if dealing with the linear case) is monoidal.

Table 8.1: Monoids, categories and deltacategories

	Type	Representations	Linearization
One-object case	monoid	monoidal category	bialgebra
Many-object case	small category	large category	algebra
	deltacategory	monoidal category	bialgebra

Before going into examples, let us say a word about a general deltacategory $(\mathfrak{C}, \Delta, \epsilon)$ in *Sets*. Let $\mathfrak{C} = (A, X, \dots)$. As explained in section 7.4, X is then a monoid with multiplication $xy = \Delta_0(x, y)$ and unit element $1 = \epsilon_0(*)$. In the notation of section 4.3, $\Delta_1 : A \times^X (X \times X) \to A \times A$ and $\epsilon_1 : A \times^X I \to I$ provide lifts of arrows as in the following pictures

$$\begin{tikzcd}
\mathfrak{C} \times \mathfrak{C} & (x,y) \arrow[d, mapsto] \arrow[r, bend left, "\Delta_1(a,(x,y))"] & \bullet \arrow[d, mapsto] & \mathfrak{I} & * \arrow[d, mapsto] \arrow[r, bend left, "\mathsf{id}_*"] & * \arrow[d, mapsto] \\
\mathfrak{C} \arrow[u, "\Delta"] & xy \arrow[r, bend left, "a"] & \bullet & \mathfrak{C} \arrow[u, "\epsilon"] & 1 \arrow[r, bend left, "a"] & 1
\end{tikzcd}\;.$$

The pictures indicate the behavior of Δ_1 and ϵ_1 with respect to source and targets (when targets are not relevant we use the symbol $\bullet$, which is not to be confused with the element $* \in I$). They must also preserve compositions and identities. Notice that ϵ_1 is uniquely determined, and automatically preserves compositions and identities. The remaining conditions are coassociativity and counitality. In most of the examples that follow these will be checked with the aid of this pictorial notation.

8.1 Double groups

Let (Γ, G) be a *double group* (also called a *matched pair of groups*, as in [K, definition IX.1.1]). Thus, there is given a left action of Γ on G, $(\gamma, g) \mapsto \gamma \cdot g$, and a right action of G on Γ, $(\gamma, g) \mapsto \gamma^g$, such that

$$\gamma \cdot (fg) = (\gamma \cdot f)(\gamma^f \cdot g) \tag{1}$$

$$(\delta\gamma)^g = \delta^{\gamma \cdot g}\gamma^g \tag{2}$$

Examples of double groups abound; the simplest one being that when Γ acts on G by automorphisms and G acts trivially on Γ.

A deltacategory $\mathfrak{C} = \mathfrak{C}(\Gamma, G)$ may be attached to every double group (Γ, G) as follows. $\mathfrak{C} = (\Gamma \times G, G, \tilde{s}, \tilde{t}, i, m)$ where

- $\tilde{s}(\gamma, g) = g$ and $\tilde{t}(\gamma, g) = \gamma \cdot g$,
- $i(g) = (1, g)$ and $m((\delta, \gamma \cdot g), (\gamma, g)) = (\delta\gamma, g)$.

In pictures:

$$\begin{tikzcd}
g \arrow[r, bend left, "{(\gamma,g)}"] \arrow[rr, bend right, "{(\delta\gamma,g)}"'] & \gamma \cdot g \arrow[r, bend left, "{(\delta,\gamma\cdot g)}"] & \delta \cdot \gamma \cdot g & & g \arrow[loop above, "{(1,g)}"] & .
\end{tikzcd}$$

Composition and identities preserve targets precisely because of the axioms for an action: $\delta \cdot \gamma \cdot g = (\delta\gamma) \cdot g$ and $1 \cdot g = g$. Associativity and unitality for m and i boil down to those of Γ.

Notice that the definition of the category structure on $\mathfrak{C}$ only involves the action of Γ on G, not the group structure of G or the action of G on Γ. These are used to define the deltacategory structure $(\mathfrak{C}, \Delta, \epsilon)$ on $\mathfrak{C}$ as follows.

- $\Delta_0 : G \times G \to G$ and $\epsilon_0 : I \to G$ are the multiplication and unit maps of the group G,
- $\Delta_1 : (\Gamma \times G) \times^G (G \times G) \to (\Gamma \times G) \times (\Gamma \times G)$ is $(\gamma, fg, f, g) \mapsto (\gamma, f, \gamma^f, g)$,
- $\epsilon_1 : (\Gamma \times G) \times^G I \to I$ is $(\gamma, 1, *) \mapsto *$.

Notice that Δ_1 is a morphism of (left) G-graded sets precisely by equation (1) above, and so is ϵ_1 because it is a consequence of (1) that $\gamma \cdot 1 = 1 \ \forall\ \gamma \in \Gamma$. It is obvious that they are morphisms of (right) $G \times G$-graded sets and I-graded sets respectively. Let us check that Δ preserves compositions and identities. To this end, let us denote the arrow (γ, g) by $g \xrightarrow{\gamma} \gamma \cdot g$. When not relevant, we may also omit the target and simply write $g \xrightarrow{\gamma} \bullet$. Below, the picture on the left shows the lifts of two composable arrows, and that on the right, the lift of their composition. We see that the composition of the lifts coincides with the lift of the composition precisely by equation (2) above.

$$\begin{tikzcd}
\mathfrak{C} \times \mathfrak{C} & (f,g) \arrow[r, bend left, "{(\gamma,\gamma^f)}"] \arrow[d, mapsto] & (\gamma \cdot f, \gamma^f \cdot g) \arrow[r, bend left, "{(\delta,\delta^{\gamma \cdot f})}"] \arrow[d, mapsto] & \bullet \arrow[d, mapsto] & & (f,g) \arrow[r, bend left, "{(\delta\gamma,(\delta\gamma)^f)}"] \arrow[d, mapsto] & \bullet \arrow[d, mapsto] \\
\mathfrak{C} \arrow[u, "\Delta"] & fg \arrow[r, bend left=10, "\gamma"] & \gamma \cdot fg \arrow[r, bend left=10, "\delta"] & \delta \cdot \gamma \cdot fg & = & fg \arrow[r, bend left=10, "\delta\gamma"] & \delta\gamma \cdot fg
\end{tikzcd}.$$

Also, the lift of the identity of fg to (f,g) is

$$\begin{tikzcd}
\mathfrak{C} \times \mathfrak{C} & (f,g) \arrow[r, bend left=10, "{(1,1^f)}"] \arrow[d, mapsto] & \bullet \arrow[d, mapsto] \\
\mathfrak{C} \arrow[u, "\Delta"] & fg \arrow[r, bend left=10, "1"] & fg
\end{tikzcd},$$

which is the identity of (f,g), since it is a consequence of (2) that $1^f = 1 \ \forall\ f \in G$.

It only remains to show that Δ and ϵ are coassociative and counital (recall that the fact that ϵ preserves compositions and identities is trivial, since $\mathfrak{I}$ is the one-arrow category). It is here where associativity and unitality of the action of G on Γ enters. Coassociativity and counitality reduce respectively to $\gamma^{fg} = (\gamma^f)^g$ and $\gamma^1 = \gamma$, as the following pictures show:

$$\begin{tikzcd}
\mathfrak{C}\times\mathfrak{C}\times\mathfrak{C} & (f,g,h) \arrow[r, bend left, "{(\gamma,\gamma^f,\gamma^{fg})}"] \arrow[d, mapsto] & \bullet \arrow[d, mapsto] \\
\mathfrak{C}\times\mathfrak{C} \arrow[u, "\Delta\times\mathsf{id}"] & (fg,h) \arrow[r, bend left, "{(\gamma,\gamma^{fg})}"] \arrow[d, mapsto] & \bullet \arrow[d, mapsto] \\
\mathfrak{C} \arrow[u, "\Delta"] & fgh \arrow[r, bend left, "\gamma"] & \gamma\cdot fgh
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{C}\times\mathfrak{C}\times\mathfrak{C} & (f,g,h) \arrow[r, bend left, "{(\gamma,\gamma^f,(\gamma^f)^g)}"] \arrow[d, mapsto] & \bullet \arrow[d, mapsto] \\
\mathfrak{C}\times\mathfrak{C} \arrow[u, "\mathsf{id}\times\Delta"] & (f,gh) \arrow[r, bend left, "{(\gamma,\gamma^f)}"] \arrow[d, mapsto] & \bullet \arrow[d, mapsto] \\
\mathfrak{C} \arrow[u, "\Delta"] & fgh \arrow[r, bend left, "\gamma"] & \gamma\cdot fgh
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{I}\times\mathfrak{C} & (*,f) \arrow[r, bend left, "{(*,\gamma^1)}"] \arrow[d, mapsto] & \bullet \arrow[d, mapsto] \\
\mathfrak{C}\times\mathfrak{C} \arrow[u, "\epsilon\times\mathsf{id}"] & (1,f) \arrow[r, bend left, "{(\gamma,\gamma^1)}"] \arrow[d, mapsto] & \bullet \arrow[d, mapsto] \\
\mathfrak{C} \arrow[u, "\Delta"] & f \arrow[r, bend left, "\gamma"] & \gamma\cdot f
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{C}\times\mathfrak{I} & (f,*) \arrow[r, bend left, "{(\gamma,*)}"] \arrow[d, mapsto] & \bullet \arrow[d, mapsto] \\
\mathfrak{C}\times\mathfrak{C} \arrow[u, "\mathsf{id}\times\epsilon"] & (f,1) \arrow[r, bend left, "{(\gamma,\gamma^f)}"] \arrow[d, mapsto] & \bullet \arrow[d, mapsto] \\
\mathfrak{C} \arrow[u, "\Delta"] & f \arrow[r, bend left, "\gamma"] & \gamma\cdot f
\end{tikzcd}$$

The monoidal category of representations of the deltacategory $\mathfrak{C}(\Gamma, G)$ admits a simple description. A representation is a set X together with a left action of Γ on X, $(\gamma, x) \mapsto \gamma\cdot x$, and a G-grading on X, $x \mapsto |x|$, such that

$$|\gamma\cdot x| = \gamma\cdot|x| \ .$$

The tensor product of two representations X and Y is $X \times Y$ with

$$\gamma\cdot(x,y) = (\gamma\cdot x, \gamma^{|x|}\cdot y) \ \text{ and } \ |(x,y)| = |x||y| \ ;$$

this follows immediately from the definition of Δ and the description of restriction along cofunctors in example 6.2.1. A particular instance of this is the category of *crossed G-sets*, that was introduced by Freyd and Yetter to construct invariants of knots [K, chapter XIV.5.2]. This is obtained from the double group (G, G) arising from the left action of G on itself by conjugation. It is possible to construct invariant of knots from more general double groups, and to explicitly describe all possible braidings on their categories of representations.

A related example. Let G be any group, and consider the category $\mathfrak{C} = (\mathbb{Z} \times G \times G, G, \dots)$ with source, target, composition and identities described by the following pictures:

$$\begin{tikzcd}
f \arrow[r, bend left, "{(n,g,f)}"] \arrow[rr, bend right, "{(m+n,hg,f)}"'] & gfg^{-1} \arrow[r, bend left, "{(m,h,gfg^{-1})}"] & hgfg^{-1}h^{-1} & f \arrow[loop above, "{(0,1,f)}"] & .
\end{tikzcd}$$

There is a deltacategory structure on $\mathfrak{C}$ as follows.

- $\Delta_0 : G \times G \to G$ and $\epsilon_0 : I \to G$ are the multiplication and unit maps of the group G,
- $\Delta_1 : (\mathbb{Z} \times G \times G) \times^G (G \times G) \to (\mathbb{Z} \times G \times G) \times (\mathbb{Z} \times G \times G)$ is $(n, x, fg, f, g) \mapsto (n, x(fg)^n f^{-n}, f, n, x(fg)^n g^{-n}, g)$,
- $\epsilon_1 : (\mathbb{Z} \times G \times G) \times^G I \to I$ is $(n, g, 1, *) \mapsto *$.

Notice that the underlying category is a particular example of those considered before (for the double group $(\mathbb{Z} \times G, G)$ arising from the action of $\mathbb{Z} \times G$ on G where $\mathbb{Z}$ and G act on G trivially and by conjugation respectively), but the deltacategory structure is different, so this is really a new example.

The verification of the deltacategory axioms is similar to the case of double groups. For instance, Δ_1 is a morphism of (left) G-graded sets because

$$\text{target of}\Delta_1(n,x,fg,f,g) = \Big(x(fg)^n f^{-n} ff^n(fg)^{-n}x^{-1} \ , x(fg)^n g^{-n} gg^n(fg)^{-n}x^{-1}\Big)$$

which maps by Δ_0 to

$$\Big(x(fg)^n f^{-n} ff^n(fg)^{-n}x^{-1}\Big)\Big(x(fg)^n g^{-n} gg^n(fg)^{-n}x^{-1}\Big) = x(fg)^n fg(fg)^{-n}x^{-1}$$
$$= xfgx^{-1} = \text{target of } (n,x,fg,f,g) \ .$$

Let us check that Δ preserves identities and compositions, but omit the verification of the remaining conditions. Let us denote the arrow (n,g,f) by $f \xrightarrow{n,g} gfg^{-1}$. Thus the lift of $fg \xrightarrow{n,x} xfgx^{-1}$ to (f,g) via Δ is

$$(f,g) \xrightarrow{n,x(fg)^n f^{-n},n,x(fg)^n g^{-n}} (x(fg)^n f(fg)^{-n}x^{-1}, x(fg)^n g(fg)^{-n}x^{-1}) \ .$$

The identity of fg is $fg \xrightarrow{0,1} fg$, and its lift to (f,g) is

$$(f,g) \xrightarrow{0,1(fg)^0 f^{-0},0,1(fg)^0 g^{-0}} \bullet = (f,g) \xrightarrow{0,1,0,1} (f,g) \ ,$$

which is the identity of the pair (f,g). The composition $fg \xrightarrow{n,x} xfgx^{-1} \xrightarrow{m,y} \bullet$ is $fg \xrightarrow{m+n,yx} \bullet$, and its lift to (f,g) is

$$(f,g) \xrightarrow{m+n,yx(fg)^{m+n} f^{-m-n},m+n,yx(fg)^{m+n} g^{-m-n}} \bullet \ .$$

On the other hand, the lift of $fg \xrightarrow{n,x} xfgx^{-1}$ to (f,g) is

$$(f,g) \xrightarrow{n,x(fg)^n f^{-n},n,x(fg)^n g^{-n}} (x(fg)^n f(fg)^{-n}x^{-1}, x(fg)^n g(fg)^{-n}x^{-1}) \ ,$$

and the succesive lift of $xfgx^{-1} \xrightarrow{m,y} \bullet$ is

$$(x(fg)^n f(fg)^{-n}x^{-1}, x(fg)^n g(fg)^{-n}x^{-1}) \xrightarrow{m, y(xfgx^{-1})^m x(fg)^n f^{-m}(fg)^{-n}x^{-1}, m, y(xfgx^{-1})^m x(fg)^n g^{-m}(fg)^{-n}x^{-1}} \bullet$$

The composition of these two lifts coincides with the lift of the composition computed above, since

$$y(xfgx^{-1})^m x(fg)^n f^{-m}(fg)^{-n}x^{-1}x(fg)^n f^{-n} = yx(fg)^{m+n}f^{-m-n}$$

and

$$y(xfgx^{-1})^m x(fg)^n g^{-m}(fg)^{-n}x^{-1}x(fg)^n g^{-n} = yx(fg)^{m+n}g^{-m-n} \ .$$

This construction of a deltacategory is actually a particular case of a more general one: it is possible to similarly associate a deltacategory to any double group for which its category of representations is braided. The category of representations of the new deltacategory is not only braided but *balanced* in a canonical way. The case presented above corresponds to the double group (G, G) whose representations are crossed G-sets.

8.2 Torsion groups

Let $\mathcal{G}$ be the groupoid with objects $\mathbb{Z}^+$ and morphisms $\mathcal{G}(n,m) = \begin{cases} \mathbb{Z}_n^* & \text{if } n = m, \\ \emptyset & \text{if } n \neq m, \end{cases}$ where $\mathbb{Z}_n^*$ denotes the group of units in the ring $\mathbb{Z}_n$ of integers modulo n. Let us use $[a]_n \in \mathbb{Z}_n$ to denote the class modulo n of $a \in \mathbb{Z}$. Thus $\mathcal{G} = (A, \mathbb{Z}^+, \tilde{s}, \tilde{t}, i, m)$ where

- $A = \coprod_{n \in \mathbb{Z}^+} \mathbb{Z}_n^*$,

- $\tilde{s}([a]_n) = n = \tilde{t}([a]_n)$,
- $i(n) = [1]_n \in \mathbb{Z}_n^*$ and $m([a]_n, [b]_n) = [ab]_n$.

There is a deltacategory structure on $\mathcal{G}$ as follows.

- $\Delta_0 : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ is $(n, m) \mapsto \mathrm{lcm}(n, m)$, the least common multiple of n and m,
- $\epsilon_0 : I \to \mathbb{Z}^+$ is $* \mapsto 1$,
- $\Delta_1 : A \times^{\mathbb{Z}^+} (\mathbb{Z}^+ \times \mathbb{Z}^+) \to A \times A$ is $([a]_{\mathrm{lcm}(n,m)}, n, m) \mapsto ([a]_n, [a]_m)$,
- $\epsilon_1 : A \times^{\mathbb{Z}^+} I \to I$ is $[1]_1 \mapsto *$.

Notice that $A \times^{\mathbb{Z}^+} I = \mathbb{Z}_1 = \{[1]_1\}$. Also, if $\gcd(a, \mathrm{lcm}(n, m)) = 1$ then $\gcd(a, n) = \gcd(a, m) = 1$, so Δ_1 is well-defined. The verification of the deltacategory axioms is trivial in this case. For instance, the lift of $[a]_{\mathrm{lcm}(n,m,l)}$ to (n, m, l) by either $(\Delta \times \mathsf{id}) \circ \Delta$ or $(\mathsf{id} \times \Delta) \circ \Delta$ is $([a]_n, [a]_m, [a]_l)$.

If G is a *torsion* group (that is, $\forall g \in G \; \exists \; n(g) \in \mathbb{Z}^+$ such that $g^{n(g)} = 1$), then G can be equipped with a natural structure of $\mathcal{G}$-representation, as follows:

- $G \to \mathbb{Z}^+$ assigns to $g \in G$ its (finite) order $|g| \in \mathbb{Z}^+$; this defines the left $\mathbb{Z}^+$-comodule structure on G,
- the action of $[a]_n \in \mathbb{Z}_n^*$ on $g \in G$ with $|g| = n$ is $[a]_n \cdot g = g^a$.

Notice that g^a is well-defined because $|g| = n$; this also implies, since $\gcd(a, n) = 1$, that $|g^a| = |g|$, which shows that the action above is a morphism of left $\mathbb{Z}^+$-comodules.

The other conditions (associativity and unitality) are clear. The $\mathcal{G}$-representation structure on $G \times H$ resulting from the deltacategory structure on $\mathcal{G}$ is just the one corresponding to the direct product structure on the group $G \times H$, because $|(g,h)| = \mathrm{lcm}(|g|,|h|)$ and $(g,h)^a = (g^a, h^a)$. If G is abelian, then it becomes a monoid in the monoidal category $\mathsf{Rep}_{\mathcal{S}}(\mathcal{G})$, because in this case $|gh| = |g||h|$ and $(gh)^a = g^a h^a$, which means that the multiplication of G is a morphism of $\mathcal{G}$-representations.

8.3 Distributive lattices

Every distributive lattice with top element yields a deltacategory structure on its underlying poset in a natural way. Before proceeding with the details of this construction, let us recall the relevant definitions. The reference for this basic material on lattices is [Grä, I.1 and I.4].

A lattice is a poset $(L, \leq)$ where every pair of elements x, y has a least upper bound and a greatest lower bound, denoted $x \vee y$ and $x \wedge y$ respectively. A lattice is thus equipped with two binary operations $\vee$ and $\wedge$ that satisfy the following properties (idempotency, commutativity, associativity and absorption):

(1) $x \vee x = x = x \wedge x$,

(2) $x \vee y = y \vee x, \qquad x \wedge y = y \wedge x$,

(3) $(x \vee y) \vee z = x \vee (y \vee z), \qquad (x \wedge y) \wedge z = x \wedge (y \wedge z)$,

(4) $x \vee (x \wedge y) = x, \qquad x \wedge (x \vee y) = x$.

The partial order may recovered from either $\vee$ or $\wedge$:

$$(5)\ \ x \leq y \Leftrightarrow x \vee y = y \Leftrightarrow x \wedge y = x.$$

A lattice satisfying any of the equivalent conditions below is called distributive [Grä, lemma I.4.10]:

$$(6)\ \ (x \vee z) \wedge (y \vee z) = (x \wedge y) \vee z \qquad \forall\, x, y, z \in L,$$

$$(6')\ \ (x \wedge z) \vee (y \wedge z) = (x \vee y) \wedge z \qquad \forall\, x, y, z \in L.$$

Now assume for a moment that $(L, \leq)$ is an arbitrary poset. An element $\hat{1} \in L$ is called a top element if $x \leq \hat{1}\ \forall\, x \in L$. A poset possesses at most one top element. Notice that it satisfies

$$(0)\ \ x \vee \hat{1} = \hat{1} \ \text{ and } \ x \wedge \hat{1} = x \ \ \forall\, x \in L\ .$$

A poset $(L, \leq)$ is viewed as a category $\mathfrak{L} = (A, L, \tilde{s}, \tilde{t}, i, m)$ as follows:

- $A = \{(x, y) \in L \times L \ /\ x \leq y\}$,
- $\tilde{s}(x, y) = x$ and $\tilde{t}(x, y) = y$,
- $i(x) = (x, x)$ and $m((y, z), (x, y)) = (x, z)$.

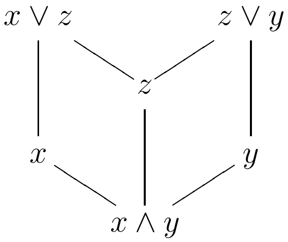
Thus, $\mathfrak{L}$ is a subcategory of the pair category $\underset{\vee}{\widehat{L}}$.

Now we are ready to describe the deltacategory structure announced above. Assume that L is a distributive lattice with top element $\hat{1}$. Let us prove that $\mathfrak{L}$ admits a deltacategory structure, as follows.

- $\Delta_0 : L \times L \to L$ is $(x, y) \mapsto x \wedge y$ and $\epsilon_0 : I \to L$ is $* \mapsto \hat{1}$,

- $\Delta_1 : A \times^L (L \times L) \to A \times A$ is $((x \wedge y, z), x, y) \mapsto ((x, x \vee z), (y, y \vee z))$,
- $\epsilon_1 : A \times^L I \to I$ is $((\hat{1}, \hat{1}), *) \mapsto *$.

The part of the diagram of L relevant to the definition of Δ_1 is



First notice that Δ_0 and ϵ_0 equip L with a monoid structure by (3) and (0). Δ_1 is a morphism of left L-graded sets, since $(x \vee z) \wedge (y \vee z) \overset{(6)}{=} (x \wedge y) \vee z \overset{(5)}{=} z$. And so is ϵ_1, because if $((x, y), *) \in A \times^L I$ then $x = \hat{1}$ and then, by definition of $\hat{1}$, $y = \hat{1}$.

Δ_1 preserves identities by (4) and (2):

$$\Delta_1((x \wedge y, x \wedge y), x, y) = ((x, x \vee (x \wedge y), (y, y \vee (x \wedge y)) = ((x, x), (y, y)) \ .$$

Consider two composable arrows $(x \wedge y, z)$ and (z, w), and their succesive lifts to (x, y) in $\mathfrak{L} \times \mathfrak{L}$, as illustrated below:

$$\begin{tikzcd}
\mathfrak{L} \times \mathfrak{L} & (x,y) \arrow[r, bend left] \arrow[d, mapsto] & (x \vee z, y \vee z) \arrow[r, bend left] \arrow[d, mapsto] & ((x \vee z) \vee w, (y \vee z) \vee w) \arrow[d, mapsto] & {} \arrow[d, phantom, "="] & (x,y) \arrow[r, bend left] \arrow[d, mapsto] & (x \vee w, y \vee w) \arrow[d, mapsto] \\
\mathfrak{L} \arrow[u, "\Delta"] & x \wedge y \arrow[r, bend left] & z \arrow[r, bend left] & w & {} & x \wedge y \arrow[r, bend left] & w
\end{tikzcd}$$

We see that the composite of the lifts coincides with the lift of the composite because $(x \vee z) \vee w = x \vee w$ and $(y \vee z) \vee w = y \vee w$ by (3) and (5), since here $z \leq w$. Thus Δ_1 preserves compositions.

It only remains to check coassociativity and counitality for Δ_1 and ϵ_1. Consider the lift of an arrow $(x \wedge y \wedge z, w)$ first by Δ to $(x \wedge y, z)$ in $\mathfrak{L} \times \mathfrak{L}$ and then by $\Delta \times \mathsf{id}$ to (x, y, z) in $\mathfrak{L} \times \mathfrak{L} \times \mathfrak{L}$, as illustrated below:

$$\begin{tikzcd}
\mathfrak{L} \times \mathfrak{L} \times \mathfrak{L} & (x, y, z) \arrow[r, bend left] \arrow[d, mapsto] & (x \vee ((x \wedge y) \vee w), y \vee ((x \wedge y) \vee w), z \vee w) \arrow[d, mapsto] \; . \\
\mathfrak{L} \times \mathfrak{L} \arrow[u, "\Delta \times \mathsf{id}"] & (x \wedge y, z) \arrow[r, bend left] \arrow[d, mapsto] & ((x \wedge y) \vee w, z \vee w) \arrow[d, mapsto] \\
\mathfrak{L} \arrow[u, "\Delta"] & x \wedge y \wedge z \arrow[r, bend left] & w
\end{tikzcd}$$

By (3) and (4) we have that $x \vee ((x \wedge y) \vee w) = (x \vee (x \wedge y)) \vee w = x \vee w$ and (using also (2)) $y \vee ((x \wedge y) \vee w) = (y \vee (x \wedge y)) \vee w = y \vee w$. Thus the lift in question is $((x, x \vee w), (y, y \vee w), (z, z \vee w))$. By symmetry this must also be the lift by $(\mathsf{id} \times \Delta) \circ \Delta$, proving coassociativity.

Finally, the lift by $(\epsilon \times \mathsf{id}) \circ \Delta$ of the arrow $(\hat{1} \wedge x, y)$ to $(*, x)$ in $\mathfrak{J} \times \mathfrak{L}$ is

$$\begin{tikzcd}
\mathfrak{J} \times \mathfrak{L} & (*, x) \arrow[r, bend left] \arrow[d, mapsto] & (*, x \vee y) \arrow[d, mapsto] \; . \\
\mathfrak{L} \times \mathfrak{L} \arrow[u, "\epsilon \times \mathsf{id}"] & (\hat{1}, x) \arrow[r, bend left] \arrow[d, mapsto] & (\hat{1} \vee y, x \vee y) \arrow[d, mapsto] \\
\mathfrak{L} \arrow[u, "\Delta"] & \hat{1} \wedge x \arrow[r, bend left] & y
\end{tikzcd}$$

By (0), $\hat{1} \wedge x = x$, hence $x \leq y$, so $x \vee y = y$ by (5). This proves left counitality, right counitality holds by the same reason.

This completes the verification of the deltacategory axioms for $(\mathfrak{L}, \Delta, \epsilon)$.

A related example. A poset may carry a deltacategory structure even if it is not a lattice. For instance, the discrete category $\widehat{\mathrm{X}}$ on a set X is a poset (but not a lattice, unless $X = \{*\}$), and a deltacategory as long as X is a monoid. For a more interesting example, consider the category

$$\mathfrak{C} = \quad 1 \xleftarrow{a} x \xrightarrow{b} y \ .$$

Notice that $\mathfrak{C}$ is the poset $X = \{1, x, y\}$ with $x \leq 1$, $x \leq y$. As such, it is not a lattice: $1 \vee y$ does not exist. However, $\mathfrak{C}$ carries a deltacategory structure, as follows.

The monoid structure on X is

	1	x	y
1	1	x	y
x	x	y	y
y	y	y	y

.

This is a submonoid of $M_3(\mathbb{N})$ via $1 = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$, $x = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, and $y = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$.

The lifting Δ_1 is described through the pictures

$$\begin{tikzcd}
\mathfrak{C} \times \mathfrak{C} & (x,1) \arrow[r, bend left, "{(a,\mathsf{id}_1)}"] \arrow[d, mapsto] & (1,1) \arrow[d, mapsto] & (1,x) \arrow[r, bend left, "{(\mathsf{id}_1,a)}"] \arrow[d, mapsto] & (1,1) \arrow[d, mapsto] \\
\mathfrak{C} \arrow[u, "\Delta"] & x \arrow[r, bend left, "a"] & 1 & x \arrow[r, bend left, "a"] & 1 \\
 & (x,1) \arrow[r, bend left, "{(b,\mathsf{id}_1)}"] \arrow[d, mapsto] & (y,1) \arrow[d, mapsto] & (1,x) \arrow[r, bend left, "{(\mathsf{id}_1,b)}"] \arrow[d, mapsto] & (1,y) \arrow[d, mapsto] \\
 & x \arrow[r, bend left, "b"] & y & x \arrow[r, bend left, "b"] & y
\end{tikzcd}$$

Together with the condition that identities must be preserved, this completely determines Δ_1. ϵ_1 is, as always, uniquely determined.

Coassociativity, counitality and preservation of compositions are in this case straightforward.

8.4 Another example

There is a way to enlarge any given category in ***Sets*** to a deltacategory, that we now describe. This construction is due to Chase.

For any set X, let $F(X) = \coprod_{n\geq 0} X^n$ denote the free monoid on X; X^n is the cartesian product of X with itself n times, $X^0 = \{*\}$. If $\mathfrak{C} = (A, X, \tilde{s}, \tilde{t}, i, m)$ is a category in ***Sets***, then $F(\mathfrak{C}) = (F(A), F(X), F(\tilde{s}), F(\tilde{t}), F(i), F(m) \circ \tau)$ is a category in ***Monoids***, where τ is the isomorphism of monoids

$$\begin{aligned} F(A) \times^{F(X)} F(A) &\xrightarrow{\tau} F(A \times^X A) \ , \\ ((a_1, \ldots, a_n), (b_1, \ldots, b_n)) &\mapsto ((a_1, b_1), \ldots, (a_n, b_n)) \ . \end{aligned}$$

(Notice that $F(A) \times F(A) \not\cong F(A \times A)$, though).

$F(\mathfrak{C})$ is also a category in ***Sets***, and can be equipped with a structure of deltacategory in ***Sets*** (not in ***Monoids***), as follows.

- $\Delta_0 : F(X) \times F(X) \to F(X)$ is the multiplication of $F(X)$,

  $((x_1, \ldots, x_p), (y_1, \ldots, y_q)) \mapsto (x_1, \ldots, x_p, y_1, \ldots, y_q)$,

- $\epsilon_0 : I \to F(X)$ is $* \mapsto * \in I = X^0$,

- $\Delta_1 : F(A) \times^{F(X)} (F(X) \times F(X)) \to F(A) \times F(A)$ is

  $((a_1, \ldots, a_{p+q}), (x_1, \ldots, x_p), (y_1, \ldots, y_q)) \mapsto ((a_1, \ldots, a_p), (a_{p+1}, \ldots, a_{p+q}))$,

- $\epsilon_1 : F(A) \times^{F(X)} I \to I$ is $(*, *) \mapsto *$.

The verification of the deltacategory axioms is straightforward.

If A is a monoid then $F(\underset{\vee}{\mathrm{A}})$ is not a one-object category, rather $F(\underset{\vee}{\mathrm{A}}) = (F(A), \mathbb{N}, \ldots)$. On the other hand, $F(\widehat{\mathrm{X}}) = \widehat{\mathrm{F(X)}}$, the deltacategory corresponding to the bimonoid $F(X)$.

Part II

Applications to quantum groups

Chapter 9

Categories in Vector Spaces

In this chapter we apply the theory of part I to study internal categories in the monoidal category Vec_k of vector spaces over a field k. The connections to quantum groups are as follows.

First, it will be shown that several basic objects in the theory of Hopf algebras or quantum groups, like Hopf modules or Yetter-Drinfeld modules, naturally appear as representations of internal categories in Vec_k (on the other hand, they cannot always be seen just as modules over an algebra). Thus, as in the set-theoretic case, representations of internal categories provide richer examples than representations of monoids or algebras. Consideration of the underlying internal categories often allows us to prove facts about their representations without dealing with the representations themselves, as in the proof of the Fundamental Theorem on Hopf modules below (corollary 9.7.1); instead, it is possible to compare the internal categories directly, either by finding functors or cofunctors between them.

The second connection has to do with one of the main goals of the theory of quantum groups; namely, the construction of monoidal categories and of bialgebras or Hopf algebras. Monoidal categories are obtained as representations of deltacategories in Vec_k, and bialgebras as admissible sections of such. In fact, we find that several important quantum groups, like Drinfeld's double and Drinfeld's and Jimbo's quantized enveloping algebras $U_q^+(g)$, are obtained as admissible sections of some very natural deltacategories. This culminates in section 9.8 with the main construction of this thesis, a generalization of the construction of the quantum groups of Drinfeld and Jimbo, involving the *binomial braids* introduced in appendix B. In particular we believe that good evidence is provided to show that the notion of a deltacategory in Vec_k is a natural and useful one.

Let us describe the contents of the various sections in more detail. In section 9.1 we show that ordinary small linear categories are examples of internal categories in Vec_k. Admissible sections and cofunctors are shown to extend notions already present in the literature for linear categories. In section 9.2, those aspects of the theory of internal categories that are particular to the case of vector spaces are developed. In particular, it is shown that the monoid of admissible sections $\Gamma(\mathfrak{C})$ of a deltacategory $\mathfrak{C}$ in Vec_k carries a structure of k-bialgebra (as long as the base coalgebra is finite-dimensional) and that there is a monoidal functor

$$\mathsf{Rep}_k\mathfrak{C} \to \mathsf{Mod}\Gamma(\mathfrak{C}) \ .$$

Here, the results of appendix A will be used. In section 9.3, the first examples of quantum groups are presented. It is shown that Sweedler's four dimensional Hopf algebra, as well as its generalization due to Taft, naturally appear as admissible sections of deltacategories. For later examples, the general considerations on free and quotient categories of section 9.4 are needed. In section 9.5 we return to the line of examples, and construct $U_q(sl_2)$ via admissible sections. This is done for Drinfeld's double in section 9.6, where we also apply the general theory of internal categories to deduce several properties of the double. In section 9.7, smash products are seen as admissible sections, and applications to Hopf modules and Hopf bimodules are presented, along the lines suggested above. Finally, in section 9.8, we describe a general procedure for constructing a quantum group $U_H^+(X)$ out of a finite-dimensional Hopf algebra H and a Yetter-Drinfeld H-module, as admissible sections of a certain deltacategory $\mathfrak{U}_H^+(X)$ in $\mathcal{Vec}_k$. The quantum groups of Drinfeld, Jimbo and Lusztig are obtained through this procedure from the simplest choice of H: group algebras $H = kG$ of cyclic groups G. In this procedure, the action of the *binomial braids* (appendix B) $b_i^{(n)}$ on the various tensor powers $X^{\otimes n}$ plays a crucial role.

When dealing with bialgebras and comodules, Sweedler's abbreviated notation will be used, as explained in appendix A.

9.1 Linear categories as internal categories

For any set X, let kX be the *group-like* coalgebra on X, so that X is a k-basis for kX, $\Delta(x) = x{\otimes}x$ and $\epsilon(x) = 1 \;\forall\; x \in X$.

The main goal of this section is to show that a small k-linear category with object set X is the same thing as a category in Vec_k with base coalgebra kX. This important observation is due to Chase.

This is an elaboration on the following basic fact [Mon, example 1.6.7]: if (M, t) is a left kX-comodule, then $M = \oplus_{x \in X} M_x$, where $M_x = \{m \in M \ / \ t(m) = x \otimes m\}$. Similarly, if (M, s, t) is a kX-kY-bicomodule then

$$M = \bigoplus_{x \in X,\ y \in Y} M_{x,y} \ \text{ where } \ M_{x,y} = \{m \in M \ / \ t(m) = x \otimes m \text{ and } s(m) = m \otimes y\} \ .$$

An X-graded k-space is a collection $\{M_x\}_{x \in X}$ of k-spaces indexed by the elements of X. A morphism of X-graded k-spaces is a collection of k-linear maps $f_x : M_x \to N_x$. X-Y-bigraded k-spaces and their morphisms are defined similarly. The above shows that there is an equivalence between the category of kX-kY-bicomodules and that of X-Y-bigraded k-spaces,that to the bicomodule (M, s, t) assigns the bigraded space with components $M_{x,y} = \{m \in M \ / \ t(m) = x \otimes m,\ s(m) = m \otimes y\}$.

A small k-linear graph is an ordinary small graph (a graph in Sets) where each Hom-set carries a structure of k-vector space. Thus, a linear graph with object set X is just an X-X-bigraded k-space, with components $M_{x,y} = \mathsf{Hom}(y, x)$ for $x, y \in X$. Recall (definition 2.3.1) that an internal graph in Vec_k with base coalgebra kX is just a kX-kX-bicomodule. It follows from the above that there is an equivalence between the category $\mathcal{G}_{kX}$ of graphs in Vec_k with base coalgebra kX and the category of small k-linear graphs with object set X.

Let M be a kX-kY-bicomodule and N a kY-kZ-one. Then, clearly, the components of the kX-kZ-bicomodule $M \otimes^{kY} N$ are

$$(M \otimes^{kY} N)_{x,z} = \bigoplus_{y \in Y} M_{x,y} \otimes N_{y,z} \ .$$

Recall (definition 2.3.1) that an internal category in Vec_k with base coalgebra kX is a monoid in $\mathcal{G}_{kX}$, with respect to $\otimes^{kX}$. It follows that such a category can be equivalently described as a small linear graph with object set X, equipped with a structure of ordinary category such that composition $\mathsf{Hom}(y,x) \times \mathsf{Hom}(z,y) \to \mathsf{Hom}(z,x)$ is k-bilinear. This is precisely the definition of a small k-linear category (as in chapter I.8 in [ML], for the case $k = \mathbb{Z}$). This proves that there is an equivalence

$$\{ \text{ small } k\text{-linear categories with object set } X \text{ } \} \cong \{ \text{ categories in } \mathit{Vec}_k \text{ over } kX \text{ } \},$$

that sends the linear category $\mathfrak{C}$ to the internal category (A, C, s, t, i, m) with

$$C = kX, \ A = \bigoplus_{x,y} \mathsf{Hom}_{\mathfrak{C}}(y,x),$$

$s(a) = a \otimes x$ and $t(a) = y \otimes a$ for $a \in \mathsf{Hom}(x,y)$,

$i(x) = id_x$, the identity arrow of x in $\mathfrak{C}$, and

$m(a \otimes b) = a \circ b$ for $a \in \mathsf{Hom}(y,z)$ and $b \in \mathsf{Hom}(x,y)$, where $\circ$ is composition in $\mathfrak{C}$.

The notion of representations for internal categories (definition 6.1.1) boils down to the usual one for linear categories; namely, a representation of a k-linear category $\mathfrak{C}$ is just a k-linear functor $\mathfrak{C} \to \mathit{Vec}_k$. In fact, if $\mathfrak{C} = (A, kX, \dots)$ is as above, then a representation of $\mathfrak{C}$ is a left kX-comodule V, equipped with an associative and unital action $a : A \otimes^{kX} V \to V$. In other words, V is an X-graded k-space equipped with associative and unital maps $\mathsf{Hom}_{\mathfrak{C}}(y,x) \otimes V_y \to V_x$. This is equivalent to giving a linear functor $\mathfrak{C} \to \mathit{Vec}_k$ that sends x to V_x and $\alpha \in \mathsf{Hom}_{\mathfrak{C}}(y,x)$ to its action $V_y \to V_x$.

Mitchell defines in [Mit, pages 33 and 51] the *matrix ring* $[\mathfrak{C}]$ of a small k-linear category $\mathfrak{C}$, which is in fact a k-algebra. Viewing $\mathfrak{C}$ as a category in Vec_k, $\mathfrak{C} = (A, kX, \dots)$, we have defined the monoid of admissible sections $\Gamma(\mathfrak{C})$ of $\mathfrak{C}$, which is in this case a k-algebra, since the monoid structure is compatible with the underlying k-linear structure on $\Gamma(\mathfrak{C}) = \mathsf{Hom}_{kX}(kX, A)$. We claim that $[\mathfrak{C}]$ is a subalgebra of $\Gamma(\mathfrak{C})$, and that $[\mathfrak{C}] = \Gamma(\mathfrak{C})$ when the object set X is finite (this was announced in section 5.2).

In fact, by definition, $[\mathfrak{C}]$ is the k-space of matrices of the form

$$[\alpha_{x,y}]_{x,y\in X} \quad \text{with} \quad \alpha_{x,y} \in \mathsf{Hom}_{\mathfrak{C}}(y,x) = A_{x,y}$$

and such that each row and column has only finitely many non-zero entries. Thus, as vector spaces,

$$[\mathfrak{C}] \cong \bigoplus_{x,y\in X} A_{x,y} = A\ .$$

Multiplication of matrices $\alpha = [\alpha_{x,y}]$ and $\beta = [\beta_{x,y}]$ is the matrix $\alpha\beta$ with entries

$$(\alpha\beta)_{x,y} = \sum_{z\in X} \alpha_{x,z} \circ \beta_{z,y},$$

where $\circ$ denotes composition in $\mathfrak{C}$:

$$\begin{tikzcd} y \arrow[r, bend left, "\beta_{z,y}"] & z \arrow[r, bend left, "\alpha_{x,z}"] & x \end{tikzcd}\ .$$

The algebra $[\mathfrak{C}]$ does not have a unit element unless X is finite, in which case the matrix δ with $\delta_{x,y} = \begin{cases} \mathsf{id}_x & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$ is the unit element.

On the other hand, from the definition of morphism of X-graded k-spaces we have that

$$\Gamma(\mathfrak{C}) = \mathsf{Hom}_{kX}(kX, A) = \prod_{y\in X} \mathsf{Hom}_k(k\{y\}, \bigoplus_{x\in X} A_{x,y}) \cong \prod_{y\in X}\bigoplus_{x\in X} A_{x,y},$$

which can be identified with the space of matrices of the form $[\alpha_{x,y}]_{x,y\in X}$ with $\alpha_{x,y} \in A_{x,y}$ and where each column "y" has finitely many non-zero entries (but rows "x" may be infinite). Thus, there is a canonical k-linear embedding

$$[\mathfrak{C}] \hookrightarrow \Gamma(\mathfrak{C}) \quad \alpha \mapsto u_\alpha$$

that views the matrix $\alpha \in [\mathfrak{C}]$ as the admissible section $u_\alpha \in \Gamma(\mathfrak{C})$ whose value on $y \in X$ is $u_\alpha(y) = \sum_{x\in X} \alpha_{x,y} \in A$. The image of this embedding consists of those admissible sections of *finite support*, that is those $u \in \Gamma(\mathfrak{C})$ such that $u(y) \neq 0$ only for finitely many $y \in X$. In particular, when X is finite this inclusion is surjective; moreover, it preserves unit elements in this case. Let us check that, in general, it preserves multiplications:

$$\begin{aligned}(u_\alpha * u_\beta)(y) &= m(u_\alpha {\otimes^{kX}} \mathsf{id}_A) t u_\beta(y) = m(u_\alpha {\otimes^{kX}} \mathsf{id}_A) t(\sum_{z\in X} \beta_{z,y}) \\ &= m(u_\alpha {\otimes^{kX}} \mathsf{id}_A)(\sum_{z\in X} z {\otimes} \beta_{z,y}) = m(\sum_{x,z\in X} \alpha_{x,z} {\otimes} \beta_{z,y}) \\ &= \sum_{x,z\in X} \alpha_{x,z} \circ \beta_{z,y} = \sum_{x\in X} (\alpha\beta)_{x,y} = u_{\alpha\beta}(y)\ .\end{aligned}$$

This completes the proof of the claim.

Mitchell's matrix ring generalizes several other important constructions: the *path algebras* of quivers of Gabriel [Gab], the *generalized triangular matrix rings* of

Chase [Cha2], and the *incidence algebras* of finite posets of Rota [Rot, GR] (see [Mit] for more details on this). The functoriality of the construction of the matrix ring of a linear category is addressed only partially in [Mit]: only functors that are the identity on objects are considered, but the more general notion of cofunctor is not discussed (such functors are particular cofunctors, see remark 4.2.1). The functoriality of the construction of the incidence algebra of a poset is investigated more carefully in [GR1, section 3.5], presumably for the reason that, for posets, functors that are the identity on objects are not very frequent (there is such a functor $P \to Q$ only if the Q-order is a refinement of the P-order). Goldman and Rota show that there is a morphism between the incidence algebras of two posets P and Q associated to the following data (called a *proper map* in [GR1]): a map $\sigma : P \to Q$ such that

(a) σ is injective,

(b) if $\sigma(p_1) \leq \sigma(p_2)$ then $p_1 \leq p_2$, and

(c) if $\sigma(p_1) \leq q \leq \sigma(p_2)$ then there is a unique $p \in P$ such that $p_1 \leq p \leq p_2$ and $\sigma(p) = q$.

Let us view the poset P as a category $\mathfrak{P} = (A_P, P, \dots)$ as in section 8.3; we reserve P for the underlying set, and $A_P = \{(p_1, p_2) \in P \times P \ / \ p_1 \leq p_2\}$. From our point of view, the above data is just a particular type of cofunctor $\varphi : k\mathfrak{Q} \to k\mathfrak{P}$ between the linearization of $\mathfrak{Q}$ and $\mathfrak{P}$, and hence the existence of an associated morphism $\Gamma(\varphi) : \Gamma(k\mathfrak{Q}) \to \Gamma(k\mathfrak{P})$ from the incidence algebra of Q to that of P is explained (proposition 5.3.1). In fact, consider the pair $\varphi = (\varphi_1, \varphi_0)$ where $\varphi_0 : kP \to kQ$ is the linear extension of σ, $\varphi_0(p) = \sigma(p)\ \forall\ p \in P$, and $\varphi_1 : kA_Q{\otimes}^{kQ}kP \to kA_P$ is the linear extension of

$$\varphi_1\Big((\sigma(p_1), q){\otimes}p_1 \Big) = \begin{cases} (p_1, p_2) & \text{if } q = \sigma(p_2) \text{ for some } p_2 \in P, \\ 0 & \text{if } q \notin \mathsf{Im}(\sigma). \end{cases}$$

Conditions (a) and (b) ensure that φ_1 is well-defined. φ preserves identities by (a) and compositions by (c), so it is indeed a cofunctor.

Finally, let us consider general functors and cofunctors between linear categories. First notice that the functor $\boldsymbol{\mathit{Sets}} \to \boldsymbol{\mathit{Coalg}}_k$, $X \mapsto kX$ is full and faithful, being left adjoint to the functor $\boldsymbol{\mathit{Coalg}}_k \to \boldsymbol{\mathit{Sets}}$ that sends a k-coalgebra C to the set $\{c \in C \ /\ \Delta_C(c) = c{\otimes}c,\ \epsilon_C(c) = 1\}$ of its group-like elements.

Let $f = (f_1, f_0) : \mathfrak{C} \to \mathfrak{D}$ be a functor between linear categories $\mathfrak{C} = (A, kX, \dots)$ and $\mathfrak{D} = (B, kY, \dots)$, in the sense of definition 4.1.1. It follows from the above that the morphism of coalgebras $f_0 : kX \to kY$ is necessarily the k-linear extension of a map $X \to Y$. Then, the morphism of kX-kX-bicomodules $f_1 : A \to {}_{f_0}B_{f_0}$, must be given by a family of k-linear maps $f_{x,y} : A_{x,y} = \mathsf{Hom}(y, x) \to \mathsf{Hom}(f_0(y), f_0(x)) = B_{f_0(y), f_0(x)}$. Preservation of identities and compositions for f translate into the obvious conditions for f_0 and $f_{x,y}$. Thus, a functor between linear categories in the sense of section 4.1 is just a k-linear functor in the sense of [ML, chapter I.8]; in particular, it is an ordinary functor between the underlying categories in $\boldsymbol{\mathit{Sets}}$, and therefore can be represented through pictures as in section 4.3.

The situation for cofunctors is different. Let $\varphi = (\varphi_1, \varphi_0) : \mathfrak{C} \to \mathfrak{D}$ be a cofunctor between linear categories as above. As before, $\varphi_0 : kY \to kX$ must be the k-linear extension of a map $Y \to X$. Then, the morphism of kX-kY-bicomodules

$\varphi_1 : A{\otimes}^{kX}kY \to {}_{\varphi_0}B$ must given by a family of k-linear maps

$$A_{x',\varphi_0(y)} \to \bigoplus_{y' \in \varphi_0^{-1}(x)} B_{y',y} \ .$$

Therefore, φ need not be a cofunctor between the underlying categories in ***Sets***; such a cofunctor would be given instead by a family of maps

$$A_{x',\varphi_0(y)} \to \coprod_{y' \in \varphi_0^{-1}(x)} B_{y',y} \ .$$

In other words, the "lift" of an arrow $a : \varphi_0(y) \to x'$ of $\mathfrak{C}$ to the object y of $\mathfrak{D}$ provided by φ_1 does not consist just of a single arrow $y \to y'$ with $\varphi_0(y') = x'$; rather, it consists of a finite sum of such arrows (which may have different targets, all mapping to x' under φ_0). With this modification, the interpretation of cofunctors as liftings of section 4.3, and the associated pictorial description, can still be used. Thus, the fact that

$$\varphi_1(a{\otimes}y) = \sum_{i=1}^{n} b_i \ \text{ for some } b_i : y \to y'_i$$

will be represented by a picture like

$$\begin{tikzcd}
\mathfrak{D} & y \arrow[d, mapsto, "\varphi_0"'] \arrow[rr, bend left, "\sum_{i=1}^{n} b_i"] & {} & y'_i \arrow[d, mapsto, "\varphi_0"] \\
\mathfrak{C} \arrow[u, "\varphi"] & \varphi_0(y) \arrow[rr, bend left, "a"] & {} \arrow[u, mapsto, "\varphi_1"'] & x'
\end{tikzcd}$$

The usage and usefulness of this notation may become clearer after the examples of later sections; for simplicity, we may omit some of the arrows from these pictures. Preservation of compositions is easily expressed in these terms: given composable arrows a_1 and a_2 as in the picture below, one lifts a_1 to y

$$\varphi_1(a_1{\otimes}y) = \sum_i b_i \ ,$$

and computes the targets y'_i of the resulting arrows, then one lifts a_2 to each of these targets

$$\varphi_1(a_2 \otimes y'_i) = \sum_j b_{i,j} \ ,$$

then the lift of $a_2 \circ a_1$ must be the composition of the lifts

$$\varphi_1\Big((a_2 \circ a_1) \otimes y\Big) = \sum_i \Big(\sum_j b_{i,j}\Big) \circ b_i \ .$$

$$\begin{tikzcd}
\mathfrak{D} & y \arrow[r, bend left, "\sum_i b_i"] \arrow[d, mapsto] & y'_i \arrow[r, bend left, "\sum_j b_{i,j}"] \arrow[d, mapsto] & \bullet \arrow[d, mapsto] & = & y \arrow[r, bend left, "\sum_i(\sum_j b_{i,j})\circ b_i"] \arrow[d, mapsto] & \bullet \arrow[d, mapsto] \; . \\
\mathfrak{C} \arrow[u, "\varphi"] & \varphi_0(y) \arrow[r, bend left, "a_1"] & x' \arrow[r, bend left, "a_2"] & \bullet & & \varphi_0(y) \arrow[r, bend left, "a_2 \circ a_1"] & \bullet
\end{tikzcd}$$

Preservation of identities has the same meaning as in the set-theoretic case: the lift of the identity of $\varphi_0(y)$ to y must be the identity of y. Compostion of cofunctors $\varphi : \mathfrak{C} \to \mathfrak{D}$ and $\psi : \mathfrak{D} \to \mathfrak{E}$ can be described as follows: the lift of an arrow a of $\mathfrak{C}$ with source $\varphi_0\psi_0(z)$ to the object z of $\mathfrak{E}$ is obtained by first lifting a to the object $\psi_0(y)$ of $\mathfrak{D}$ by means of φ and then lifting all the obtained arrows to z by means of ψ.

9.2 Representations and admissible sections

Let $\mathfrak{C} = (A, C, s, t, i, m)$ be a category in Vec_k. The monoid of admissible sections $\Gamma(\mathfrak{C}) = \mathsf{Hom}_C(C, A)$ is a k-vector subspace of $\mathsf{Hom}_k(C, A)$, and with respect to this structure, multiplication of admissible sections is k-bilinear, since it is given by the formula (proposition 5.1.1)

$$u * v = m(u \otimes^{C} \mathsf{id}_A)tv, \quad \text{for} \quad u, v \in \Gamma(\mathfrak{C}) \ .$$

Therefore, $\Gamma(\mathfrak{C})$ is a k-algebra.

We will use $\mathsf{Rep}_k \mathfrak{C}$ to denote the category of representations of $\mathfrak{C}$ (section 6.1), instead of $\mathsf{Rep}_{\mathcal{V}ec_k} \mathfrak{C}$. In this section we will show that there is a canonical functor

$$\mathsf{Rep}_k \mathfrak{C} \to \mathsf{Mod}\Gamma(\mathfrak{C})$$

and discuss some conditions under which this functor is an equivalence. We will also show that, if $\mathfrak{C}$ is a deltacategory and C is finite-dimensional, then $\Gamma(\mathfrak{C})$ carries a structure of k-bialgebra, and the functor above preserves the resulting monoidal structures.

Since $\Gamma(\mathfrak{C})$ is a k-algebra, we can consider the *one-object* category $\underset{\smile}{\Gamma}(\mathfrak{C})$ (examples 2.4.1).

Proposition 9.2.1. *There is a cofunctor* $\varphi : \underset{\smile}{\Gamma}(\mathfrak{C}) \to \mathfrak{C}$ *defined by* $\varphi = (e, \epsilon_C)$, *where* $\epsilon_C : C \to k$ *is the counit of* C *and* $e : \Gamma(\mathfrak{C}) \otimes C \to A$ *is the evaluation map* $u \otimes c \mapsto u(c)$.

Proof. We check the conditions in definition 4.2.1. The counit ϵ_C is a morphism of coalgebras. The evaluation map e is a morphism of right C-comodules: for $u \otimes c \in \Gamma(\mathfrak{C}) \otimes C$ we have, since u is a morphism of right C-comodules,

$$se(u \otimes c) = su(c) = (u \otimes \mathsf{id}_C)\Delta_C(c) = u(c_1) \otimes c_2 = (e \otimes \mathsf{id}_C)(u \otimes c_1 \otimes c_2) = (e \otimes \mathsf{id}_C)(\mathsf{id} \otimes \Delta_C)(u \otimes c),$$

from where $se = (e \otimes \mathsf{id}_C)(\mathsf{id} \otimes \Delta_C)$ as required.

The remaining conditions are preservation of identities and compositions, i.e. the commutativity of the following diagrams:

$$\begin{tikzcd}
\Gamma(\mathfrak{C}){\otimes}C \arrow[r, "e"] & A \\
k{\otimes}C \arrow[u, "u_{\Gamma(\mathfrak{C})}\otimes\mathsf{id}_C"] & C \arrow[l, "\cong"'] \arrow[u, "i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(\mathfrak{C}){\otimes}\Gamma(\mathfrak{C}){\otimes}C \arrow[r, "\mathsf{id}\otimes e"] \arrow[d, "m\otimes^C\mathsf{id}_C"'] & \Gamma(\mathfrak{C}){\otimes}A \arrow[r, "\mathsf{id}\otimes t"] & \Gamma(\mathfrak{C}){\otimes}C{\otimes^C}A \arrow[r, "e\otimes^C\mathsf{id}_A"] & A{\otimes^C}A \arrow[d, "m"] \\
\Gamma(\mathfrak{C}){\otimes}C \arrow[rrr, "e"'] & & & A
\end{tikzcd} .$$

Evaluating on $u{\otimes}v{\otimes}c$ we see that the second diagram commutes precisely by definition of multiplication of admissible sections. In the first diagram, $u_{\Gamma(\mathfrak{C})}$ denotes the map that sends $1 \in k$ to the unit element of $\Gamma(\mathfrak{C})$, namely the admissible section i; thus, it commutes trivially. This completes the proof. $\square$

Corollary 9.2.1. *There is a functor $\mathsf{Rep}_k\mathfrak{C} \to \mathsf{Mod}\Gamma(\mathfrak{C})$, preserving the forgetful functors to Vec_k.*

Proof. This is the result of section 6.2, applied to the cofunctor φ. $\square$

Remark 9.2.1. By definition of restriction along cofunctors (section 6.2), the action of $\Gamma(\mathfrak{C})$ on a representation (X, p, a) of $\mathfrak{C}$ is

$$\Gamma(\mathfrak{C}){\otimes}X \xrightarrow{\mathsf{id}\otimes p} \Gamma(\mathfrak{C}){\otimes}C{\otimes^C}X \xrightarrow{e\otimes^C\mathsf{id}_X} A{\otimes^C}X \xrightarrow{a} X .$$

On the other hand, by proposition 6.4.1, for each representation X of $\mathfrak{C}$ there is a morphism of monoids

$$\gamma_X : \Gamma(\mathfrak{C}) \to \mathsf{End}_{\mathcal{S}}(X), \text{ such that } \gamma_X(u) : X \xrightarrow{p} C{\otimes^C}X \xrightarrow{u\otimes^C\mathsf{id}_X} A{\otimes^C}X \xrightarrow{a} X .$$

This endows X again with the same structure of $\Gamma(\mathfrak{C})$-module.

We denote by $\overleftarrow{\mathit{Cat}}_f$ the full subcategory of $\overleftarrow{\mathit{Cat}}_{\mathit{Vec}_k}$ consisting of those categories $\mathfrak{C} = (A, C, \dots)$ in Vec_k whose underlying k-coalgebra C is finite-dimensional (but A may be infinite-dimensional); we call these categories *finite*. A linear category is finite precisely when it has finitely-many objects.

The results of appendix A will be used for what follows. Let C be a finite-dimensional k-coalgebra. Then, by lemma A.2.1, C^* is a C-C-bicomodule.

Proposition 9.2.2. *Let $\mathfrak{C} = (A, C, \dots)$ be a finite category in Vec_k. Then the map*

$$d : A \otimes^C C^* \to \Gamma(\mathfrak{C}), \quad d(\sum_i a_i \otimes f_i)(c) = \sum_i f_i(c) a_i \ ,$$

is an isomorphism of C-C-bicomodules (with respect to the structures described in section A.2).

Proof. This is a particular case of lemma A.2.3, since $\Gamma(\mathfrak{C}) = \mathsf{Hom}^r_C(C, A)$. $\square$

Remark 9.2.2. A finite-dimensional k-coalgebra is called *co-Frobenius* if the dual k-algebra is Frobenius, or equivalently if C^* is isomorphic to C as left C-comodules [Doi1, section 2.1]. Any group-like coalgebra is co-Frobenius, and so is any finite-dimensional Hopf algebra [Mon, 2.1.3]. If C is co-Frobenius then proposition 9.2.2 says that $\Gamma(\mathfrak{C}) \cong A$. This was already observed for the case of group-like coalgebras in section 9.1.

Recall from section 7.4 that the admissible sections functor

$$\Gamma : \overleftarrow{\mathit{Cat}}_{\mathit{Vec}_k} \to \mathit{Alg}_k$$

together with the natural morphism of k-algebras

$$\Gamma(\mathfrak{C}) \otimes \Gamma(\mathfrak{D}) \to \Gamma(\mathfrak{C} \otimes \mathfrak{D})$$

of section 7.2 (which is given simply by the tensor product of linear maps) define a lax monoidal functor $\overleftarrow{\mathtt{Cat}}_{\mathtt{Vec}_k} \to \mathtt{Alg}_k$.

As an immediate consequence of the above proposition we obtain:

Proposition 9.2.3. *The functor* $\Gamma : \overleftarrow{\mathtt{Cat}}_f \to \mathtt{Alg}_k$ *equipped with the natural transformation* $\Gamma(\mathfrak{C}){\otimes}\Gamma(\mathfrak{D}) \to \Gamma(\mathfrak{C}{\otimes}\mathfrak{D})$ *is a weak monoidal functor.*

Proof. We only need to show that $\Gamma(\mathfrak{C}){\otimes_k}\Gamma(\mathfrak{D}) \to \Gamma(\mathfrak{C}{\otimes}\mathfrak{D})$ is an isomorphism. This follows from the commutativity of the following diagram:

$$\begin{tikzcd}
\Gamma(\mathfrak{C}){\otimes}\Gamma(\mathfrak{D}) \arrow[rr] & & \Gamma(\mathfrak{C}{\otimes}\mathfrak{D}) \\
(A{\otimes^C}C^*){\otimes}(B{\otimes^D}D^*) \arrow[u, "\cong"] \arrow[rr, "\tau_{(A,C^*,B,D)}"] & & (A{\otimes}B){\otimes^{C\otimes D}}(C^*{\otimes}D^*) \arrow[u, "\cong"']
\end{tikzcd} ,$$

where the vertical isomorphisms are those of proposition 9.2.2 (we identify $C^*{\otimes}D^* = (C{\otimes}D)^*$) and the bottom map is the isomorphism of lemma 7.1.1. This diagram indeed commutes because so does the following, clearly:

$$\begin{tikzcd}
\mathsf{Hom}_k(C,A){\otimes}\mathsf{Hom}_k(D,B) \arrow[r] & \mathsf{Hom}_k(C{\otimes}D, A{\otimes}B) \\
(A{\otimes}C^*){\otimes}(B{\otimes}D^*) \arrow[u, "\cong"] \arrow[r, "\mathsf{id}_A\otimes\tau_{C^*,B}\otimes\mathsf{id}_{D^*}"] & (A{\otimes}B){\otimes}(C^*{\otimes}D^*) \arrow[u, "\cong"']
\end{tikzcd} .$$

It follows that $\Gamma(\mathfrak{C}){\otimes_k}\Gamma(\mathfrak{D}) \to \Gamma(\mathfrak{C}{\otimes}\mathfrak{D})$ is an isomorphism and the proof is complete. ☐

Corollary 9.2.2. *If* $\mathfrak{C}$ *is a finite deltacategory, then* $\Gamma(\mathfrak{C})$ *is a* k*-bialgebra.*

Proof. Comonoids are preserved under weak monoidal functors, so the result follows from proposition 9.2.3. ☐

Notice that, moreover, the functor $\mathsf{res}_{\varphi} : \mathsf{Rep}_k\mathfrak{C} \to \mathsf{Mod}\Gamma(\mathfrak{C})$ of corollary 9.2.1 preserves the monoidal structures on these categories. This is simply because the cofunctor $\varphi : \underset{\vee}{\Gamma}(\mathfrak{C}) \to \mathfrak{C}$ preserves the deltacategory structures, as one may easily check.

We turn back our attention to the functor $\mathsf{Rep}_k\mathfrak{C} \to \mathsf{Mod}\Gamma(\mathfrak{C})$, for the case of finite categories. *Flat* comodules are defined in section A.1. Their relevance to the notions we are discussing is explained by next result. In its proof we will make use of the following fact: if $A \to B$ is a morphism of k-algebras and X is a left B-module, then the action map $B{\otimes}X \to X$ factors through $B{\otimes_A}X \to X$, where X and B are viewed as A-modules by restriction along $A \to B$. This is an obvious consequence of the associativity of the action.

Proposition 9.2.4. *If $\mathfrak{C} = (A, C, \dots)$ is a finite deltacategory and A is flat as right C-comodule, then the functor $\mathsf{Rep}_k\mathfrak{C} \to \mathsf{Mod}\Gamma(\mathfrak{C})$ is an equivalence.*

Proof. According to lemma A.2.1, we may identify $\mathsf{Comod}C = \mathsf{Mod}C^{*op}$ (left comodules and left modules). Thus, the isomorphism of C-C-bicomodules $\Gamma(\mathfrak{C}) \cong A{\otimes^{C}}C^{*}$ of proposition 9.2.2 can also be seen as an isomorphism $\Gamma(\mathfrak{C}) \cong A{\otimes^{C}}C^{*op}$ of C^{*op}-C^{*op}-bimodules, and we have an isomorphism of functors $\mathsf{Comod}C \to \mathsf{Comod}C$ as follows:

$$\Gamma(\mathfrak{C}){\otimes_{C^{*op}}}(-) \cong (A{\otimes^{C}}C^{*op}){\otimes_{C^{*op}}}(-) \cong A{\otimes^{C}}\Big(C^{*op}{\otimes_{C^{*op}}}(-)\Big) \cong A{\otimes^{C}}(-)\ .$$

(The isomorphism in the middle is that of lemma A.2.4; here is where the flatness assumption is used).

Recall (examples 4.1.1) that there is a functor $(i, \mathsf{id}_C) : \widehat{\mathrm{C}} \to \mathfrak{C}$. By remark 4.2.1, this can also be seen as a cofunctor. We know from section 5.2 that $\Gamma(\widehat{\mathrm{C}}) = C^{*op}$. Hence, by proposition 5.3.1, there is a corresponding morphism of k-algebras

$$C^{*op} = \Gamma(\widehat{\mathrm{C}}) \xrightarrow{\bar{i}} \Gamma(\mathfrak{C}) \ .$$

Explicitly, $\bar{i}$ sends $f \in C^*$ to $(f \otimes i)\Delta_C \in \Gamma(\mathfrak{C})$. Restriction along $\bar{i}$ allows us to view left or right $\Gamma(\mathfrak{C})$-modules as left or right C^{*op}-modules (or C-comodules). In particular, one checks easily that the resulting structure of right C^{*op}-module on $\Gamma(\mathfrak{C})$ is the same as the one previously considered, namely

$$u \cdot f = (f \otimes u)\Delta_C \ \ \forall f \in C^{*op}, \ u \in \Gamma(\mathfrak{C}) \ .$$

It follows that if X is a left $\Gamma(\mathfrak{C})$-module, then it is also a left C^{*op}-module (or left C-comodule) and that the structure map $\Gamma(\mathfrak{C}) \otimes X \to X$ factors through

$$\Gamma(\mathfrak{C}) \otimes_{C^{*op}} X \to X \ ,$$

(according to the remark preceding the proposition). Together with the isomorphism of functors above, this allows us to endow X with an action $A \otimes^C X \to X$, which will be associative and unital because so is the action of $\Gamma(\mathfrak{C})$. Thus X becomes a left $\mathfrak{C}$-representation, and we have constructed a functor

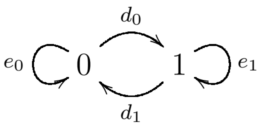
$$\mathsf{Mod}\Gamma(\mathfrak{C}) \to \mathsf{Rep}_k\mathfrak{C} \ .$$

Routine verifications show that this is the desired inverse. ☐

Remark 9.2.3. The flatness hypothesis in proposition 9.2.4 can be substituted by others. For instance, still assuming that C is finite-dimensional, but with no assumptions on A, the functor $\mathsf{Rep}_k\mathfrak{C} \to \mathsf{Mod}\Gamma(\mathfrak{C})$ is an equivalence, provided that one restricts attention to those $\mathfrak{C}$-representations and $\Gamma(\mathfrak{C})$-modules whose underlying C-comodule is flat as C^{*op}-module. In fact, the above proof works in this case too, because lemma A.2.4 can still be applied under this alternative hypothesis.

9.3 Sweedler's algebra as admissible sections

Consider the following (small) graph in $\mathcal{S}ets$



Let k be a field, $\mathsf{char} k \neq 2$. Linearizing the set of arrows we obtain a k-linear graph with object set $\mathbb{Z}_2 = \{0, 1\}$. We define a k-linear category structure on this graph by letting e_0 and e_1 be the identities, setting $d_0 \circ d_1 = d_1 \circ d_0 = 0$, and extending composition linearly. As explained in section 9.1 this yields a category $\mathfrak{C} = (A, C, \dots)$ in $\mathcal{V}ec_k$, where $A = ke_0 \oplus ke_1 \oplus kd_0 \oplus kd_1$ and $C = k\mathbb{Z}_2 = k[c]/(c^2 - 1)$. It turns out that $\mathfrak{C}$ is actually a deltacategory as follows:

$\Delta : \mathfrak{C} \to \mathfrak{C} \otimes \mathfrak{C}$ is the cofunctor $\Delta = (\Delta_1, \Delta_0)$ given by

$$\begin{array}{rcl} \Delta_0 : C \otimes C & \to & C \\ c^i \otimes c^j & \mapsto & c^{i+j} \end{array} \quad \text{and} \quad \begin{array}{rcl} \Delta_1 : A \otimes^C (C \otimes C) & \to & A \otimes A \\ e_{i+j} \otimes (c^i \otimes c^j) & \mapsto & e_i \otimes e_j \\ d_{i+j} \otimes (c^i \otimes c^j) & \mapsto & d_i \otimes e_j + (-1)^i e_i \otimes d_j \end{array}$$

(where the indices i and j are taken modulo 2) and $\epsilon : \mathfrak{C} \to \mathfrak{I}$ is the cofunctor $\epsilon = (\epsilon_1, \epsilon_0)$ given by

$$\begin{array}{rcl} \epsilon_0 : I & \to & C \\ 1 & \mapsto & c^0 \end{array} \quad \text{and} \quad \begin{array}{rcl} \epsilon_1 : A{\otimes}^{C} I & \to & I \\ e_0 & \mapsto & 1 \\ d_0 & \mapsto & 0 \end{array}$$

(recall the description of tensor coproducts over group-like coalgebras from section 9.1). By construction, Δ preserves identities. Let us check that Δ preserves compositions, but omit the verification of the other conditions in definition 7.4.1 of deltacategories. Compositions with identities are trivially preserved; the only relevant composition to be considered is $d_{i+1} \circ d_i = 0$ for $i = 0, 1 \mod 2$. We use the terminology and notation explained in section 9.1. We need to show that the lift of $d_{i+j+1}{\otimes}d_{i+j}$ to (i, j) by Δ is equal to 0. We first compute the lift of d_{i+j} to (i, j). This involves various arrows with different targets. We then compute the lifts of d_{i+j+1} to these targets, as below:

$$\begin{tikzcd}
(i,j) \arrow[d, mapsto] & \begin{array}{c} d_i{\otimes}e_j + \\ +(-1)^i e_i{\otimes}d_j \end{array} & \begin{array}{c} (i+1,j) \\ (i,j+1) \end{array} \arrow[d, mapsto] & \begin{array}{c} d_{i+1}{\otimes}e_j + (-1)^{i+1} e_{i+1}{\otimes}d_j \\ d_i{\otimes}e_{j+1} + (-1)^i e_i{\otimes}d_{j+1} \end{array} \\
i+j & d_{i+j} \arrow[u, mapsto] & i+j+1 & d_{i+j+1} \arrow[u, mapsto]
\end{tikzcd} .$$

Since Δ preserves compositions, composing the top rows we find the lift of $d_{i+j+1} \circ d_{i+j}$ to (i, j); it is

$$(d_{i+1} \circ d_i){\otimes}e_j + (-1)^{i+1} d_i{\otimes}d_j + (-1)^i \Big(d_i{\otimes}d_j + (-1)^i e_i{\otimes}(d_{j+1} \circ d_j \Big) = 0 \ ,$$

as required.

Since $\mathfrak{C}$ is a finite deltacategory, $\Gamma(\mathfrak{C})$ is a k-bialgebra (corollary 9.2.2). Let us describe it. A general admissible section $u : C \to A$ is of the form $u(c^j) = u_j e_j + u'_j d_j$ for some scalars $u_j, u'_j \in k$, for $j = 0, 1$ (a morphism of $k\mathbb{Z}_2$-comodules is a morphism of $\mathbb{Z}_2$-graded spaces, see section 9.1). Following the definition in proposition 5.1.1 we find that the multiplication in $\Gamma(\mathfrak{C})$ is described by

$$(u * v)(c^j) = u_j v_j e_j + (u'_j v_j + u_{j+1} v'_j) d_j \ .$$

Let $x : C \to A$ be $x(c^j) = d_j$ and $g : C \to A$ be $g(c^j) = (-1)^j e_j$ for $j = 0, 1$. Recall that identities are given by $i(c^j) = e_j$, $j = 0, 1$. It follows that $\{i = 1, x, g, xg\}$ is a k-basis for $\Gamma(\mathfrak{C})$ (here we use $\mathsf{char} k \neq 2$) such that

$$x^2 = 0, \ g^2 = 1 \text{ and } xg = -gx \ .$$

Let us compute $\Gamma(\Delta)(u)$, for $u = x, g$, following the definition in section 5.3:

$$\begin{tikzcd}
C{\otimes}C \arrow[r, "\Delta_{C\otimes C}"] & (C{\otimes}C){\otimes^C}(C{\otimes}C) \arrow[r, "\Delta_0\otimes^C\mathsf{id}"] & C{\otimes^C}(C{\otimes}C) \arrow[r, "u\otimes^C\mathsf{id}"] & A{\otimes^C}(C{\otimes}C) \arrow[r, "\Delta_1"] & A{\otimes}A \\
c^i{\otimes}c^j \arrow[r, mapsto] & c^i{\otimes}c^j{\otimes}c^i{\otimes}c^j \arrow[r, mapsto] & c^{i+j}{\otimes}c^i{\otimes}c^j \arrow[r, mapsto, "x\otimes^C\mathsf{id}"] & d_{i+j}{\otimes}c^i{\otimes}c^j \arrow[r, mapsto] & d_i{\otimes}e_j + (-1)^i e_i{\otimes}d_j \\
 & & & & = (x{\otimes}1 + g{\otimes}x)(c^i{\otimes}c^j), \\
c^i{\otimes}c^j \arrow[r, mapsto] & c^i{\otimes}c^j{\otimes}c^i{\otimes}c^j \arrow[r, mapsto] & c^{i+j}{\otimes}c^i{\otimes}c^j \arrow[r, mapsto, "g\otimes^C\mathsf{id}"] & (-1)^{i+j} e_{i+j}{\otimes}c^i{\otimes}c^j \arrow[r, mapsto] & (-1)^{i+j} e_i{\otimes}e_j \\
 & & & & = (g{\otimes}g)(c^i{\otimes}c^j).
\end{tikzcd}$$

Thus $\Gamma(\Delta)(x) = x{\otimes}1 + g{\otimes}x$ and $\Gamma(\Delta)(g) = g{\otimes}g$. Similarly $\Gamma(\epsilon)(x) = 0$ and $\Gamma(\epsilon)(g) = 1$. Therefore, $\Gamma(\mathfrak{C}) = H_4$, Sweedler's 4-dimensional Hopf algebra, as described in [Mon, 1.5.6].

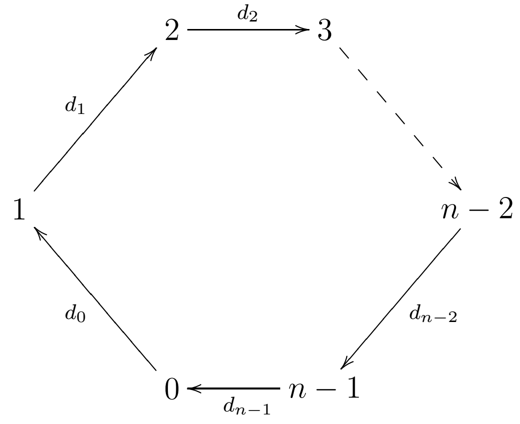
There is a generalization of Sweedler's example due to Taft; namely, for each primitive n-th root of unity q the author defines in [Taf] a k-algebra $T_n(q)$ with generators x and y subject to the relations

$$x^n = 0, y^n = 1 \ \text{ and } \ yx = qxy,$$

and proves that it is a Hopf algebra with

$$\Delta(y) = y{\otimes}y, \Delta(x) = x{\otimes}1 + y{\otimes}x, \epsilon(y) = 1 \text{ and } \epsilon(x) = 0 \ .$$

Sweedler's algebra is $T_2(-1)$. Again, $T_n(q)$ can be obtained as admissible sections of a naturally defined deltacategory in $\boldsymbol{Vec}_k$, as follows. Consider the free k-linear category on the *cyclic* graph on n vertices



modulo the relations

$$d_{i+n-1} \circ d_{i+n-2} \circ \ldots \circ d_{i+1} \circ d_i = 0 \ \text{ for } i = 0, 1, \ldots, n-1 \quad \text{mod } n.$$

Let $\mathcal{T}_n(q) = (A, C, \ldots)$ denote the corresponding category in $\boldsymbol{Vec}_k$, so that, in particular, $C = k\mathbb{Z}_n = k[c]/(c^n - 1)$ and $d_i, e_i \in A$, where e_i denotes the identity of the object i. Then $\mathcal{T}_n(q)$ carries a deltacategory structure defined on the generating arrows as follows (this claim can be checked using the general considerations on quotient categories in section 9.4 below; it also follows from the results of section 9.8, since $\mathcal{T}_n(q)$ is a particular *binomial deltacategory*, as explained in section 9.8.3):

$$\begin{array}{rcl} \Delta_0 : C \otimes C & \to & C \\ c^i \otimes c^j & \mapsto & c^{i+j} \end{array} \quad \text{and} \quad \begin{array}{rcl} \Delta_1 : A \otimes^C (C \otimes C) & \to & A \otimes A \\ e_{i+j} \otimes (c^i \otimes c^j) & \mapsto & e_i \otimes e_j \\ d_{i+j} \otimes (c^i \otimes c^j) & \mapsto & d_i \otimes e_j + q^i e_i \otimes d_j \end{array}$$

and

$$\begin{array}{rcl} \epsilon_0 : I & \to & C \\ 1 & \mapsto & c^0 \end{array} \quad \text{and} \quad \begin{array}{rcl} \epsilon_1 : A \otimes^C I & \to & I \\ e_0 & \mapsto & 1 \\ d_0 & \mapsto & 0 \end{array} .$$

Let us check in detail that $T_n(q) \cong \Gamma(\mathcal{T}_n(q))$ as bialgebras. First, consider the admissible sections $X : C \to A$, $X(c^i) = d_i$ and $Y : C \to A$, $Y(c^i) = q^i e_i$. Then, as in Sweedler's example, one checks easily that

$$X^n = 0, Y^n = 1 \quad \text{and} \quad YX = qXY,$$

$$\Gamma(\Delta)(Y) = Y \otimes Y, \Gamma(\Delta)(X) = X \otimes 1 + Y \otimes X, \Gamma(\epsilon)(Y) = 1 \text{ and } \Gamma(\epsilon)(X) = 0 \ .$$

Hence, there is a morphism of bialgebras $T_n(q) \to \Gamma(\mathcal{T}_n(q))$ sending x to X and y to Y. The point is to show that this map is bijective. Notice that $\{x^i y^j \ / \ 0 \leq i, j \leq n-1\}$ is a k-basis for $T_n(q)$, so $\dim T_n(q) = n^2$. On the other hand, recall from section 9.1 that $\Gamma(\mathcal{T}_n(q)) \cong A$ as k-spaces, and

$$\{e_i, d_i, d_{i+1} \circ d_i, \ldots, d_{i+n-2} \circ \ldots \circ d_{i+1} \circ d_i \ / \ i = 0, 1, \ldots, n-1 \mod n\}$$

is a k-basis for A, from where $\dim \Gamma(\mathcal{T}_n(q)) = n^2$ as well. (The fact that this is indeed a k-basis is clear from the definition of free k-linear category; for the more general notion of free categories in $\mathcal{V}ec_k$ see section 9.4 below). Thus, it is enough to prove that the map above is surjective. We will show that X and Y generate the k-basis of $\Gamma(\mathcal{T}_n(q))$ corresponding to the above basis of A. For each $i = 0, 1, \ldots n-1$, consider the admissible section $\delta_i : C \to A$, $\delta_i(c^j) = \delta_{ij} e_i$, where δ_{ij} is Kronecker's delta. We have a system of equations

$$\begin{cases} 1 = & \delta_0 + \delta_1 + \ldots + \delta_{n-1} \\ Y = & \delta_0 + q\delta_1 + \ldots + q^{n-1}\delta_{n-1} \\ \vdots & \\ Y^{n-1} = & \delta_0 + q^{n-1}\delta_1 + \ldots + q^{(n-1)(n-1)}\delta_{n-1} \end{cases}$$

Since q is a primitive n-th root of unity, the Vandermonde matrix $[q^{ij}]_{0 \leq i,j \leq n-1}$ is invertible. Therefore, each δ_i belongs to the subalgebra generated by Y. Notice that δ_i is the basis element of $\Gamma(\mathcal{T}_n(q))$ corresponding to $e_i \in A$. The basis element corresponding to $d_{i+j} \circ \ldots \circ d_{i+1} \circ d_i \in A$ is just $X^{j+1}\delta_i$ (computing the product of these admissible sections we find that its value on c^h is $\delta_{ih} d_{i+j} \circ \ldots \circ d_{i+1} \circ d_i$). This completes the proof.

The Hopf algebra $T_n(q)$ was also considered by Pareigis in [P2]. There it is shown that the category of left $T_n(q)$-modules admits a nice simple description in terms of *cyclic complexes*, as follows. Given a diagram of vector spaces

$$\begin{tikzcd}
& V_2 \arrow[r, "d_2"] & V_3 \arrow[dr, dashed] & \\
V_1 \arrow[ur, "d_1"] & & & V_{n-2} \arrow[dl, "d_{n-2}"] \\
& V_0 \arrow[ul, "d_0"] & V_{n-1} \arrow[l, "d_{n-1}"] &
\end{tikzcd}$$

such that

$$d_{i+n-1} \circ d_{i+n-2} \circ \ldots \circ d_{i+1} \circ d_i = 0 \quad \text{for } i = 0, 1, \ldots , n-1 \quad \bmod n,$$

the vector space $V = \oplus_{i=0}^{n-1} V_i$ carries a left $T_n(q)$-module structure, where $x \in T_n(q)$ acts on $V_i \subseteq V$ as the map d_i and $y \in T_n(q)$ acts on V_i by multiplication by q^i. Conversely, every left $T_n(q)$-module arises in this way from a cyclic complex. This description of $\mathsf{Mod} T_n(q)$ can be obtained immediately from proposition 9.2.4 (recall that the flatness hypothesis are always satisfied in the case of linear categories):

$$\mathsf{Rep}_k \mathcal{T}_n(q) \cong \mathsf{Mod}\Gamma(\mathcal{T}_n(q)) \cong \mathsf{Mod} T_n(q),$$

plus the description of representations of a linear category in section 9.1. The delta-category structure on $\mathcal{T}_n(q)$ (or the bialgebra structure on $T_n(q)$) induces a monoidal structure on $\mathsf{Rep}_k \mathcal{T}_n(q)$. This is the natural tensor product of cyclic complexes, as considered by Pareigis.

9.4 Free and quotient categories

In this section we construct the free category on a graph. A priori, “free” could be understood in two different ways: with respect to functors or cofunctors. Surprisingly, both universal problems have the same solution.

We also introduce the notion of ideal of a category and coideal of a deltacategory, and discuss the corresponding quotient constructions.

First, let us mention that the notions of functor and cofunctor can be defined for arbitrary graphs instead of categories, by omitting the associativity and unitality conditions in definitions 4.1.1 and 4.2.1.

Let $\mathcal{G} = (M, C, s, t)$ be a graph in $\boldsymbol{\mathit{Vec}}_k$, that is M is a C-C-bicomodule via t and s. Consider the C-C-bicomodule

$$\perp^C(M) = C \oplus M \oplus (M \otimes^C M) \oplus (M \otimes^C M \otimes^C M) \oplus \dots$$

(recall from section A.3 that the direct sum of bicomodules carries a natural structure of bicomodule, and that direct sums commute with tensor coproducts). Let us abbreviate $M^{\otimes^C_n} = \begin{cases} C \text{ if } n = 0 \\ M \otimes^C M^{\otimes^C_{n-1}} \text{ if } n \geq 1 \end{cases}$, so that $\perp^C(M) = \bigoplus_{n=0}^{\infty} M^{\otimes^C_n}$. Let $i : C \to \perp^C(M)$ be the canonical inclusion and $m : \perp^C(M) \otimes^C \perp^C(M) \to \perp^C(M)$ be such that its component $M^{\otimes^C_n} \otimes^C M^{\otimes^C_m} \to \perp^C(M)$ is the inclusion $M^{\otimes^C_n} \otimes^C M^{\otimes^C_m} = M^{\otimes^C_{n+m}} \subset \perp^C(M)$. Then, obviously,

$$\mathcal{T}(\mathcal{G}) = (\perp^C(M), C, s, t, i, m)$$

is a category in $\boldsymbol{\mathit{Vec}}_k$. Moreover, there is a canonical functor $\overrightarrow{j} = (j, \mathsf{id}_C) : \mathcal{G} \to$

$\mathcal{T}(\mathcal{G})$, where $j : M \hookrightarrow \perp^C(M)$ is the inclusion. This is the free category on the graph $\mathcal{G}$, in the sense that the following universal property holds.

Proposition 9.4.1. *Given any category* $\mathfrak{D}$ *and a functor (of graphs)* $f : \mathcal{G} \to \mathfrak{D}$, *there is a unique functor (of categories)* $\tilde{f} : \mathcal{T}(\mathcal{G}) \to \mathfrak{D}$ *making the following diagram commutative*

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\vec{j}"] \arrow[dr, "f"'] & \mathcal{T}(\mathcal{G}) . \arrow[d, dashed, "\tilde{f}"] \\
& \mathfrak{D}
\end{tikzcd}$$

Proof. The proof is straightforward but we sketch the details. Let $\mathfrak{D} = (B, D, \dots)$ and $f = (f_1, f_0)$. Thus $f_0 : C \to D$ is a morphism of coalgebras and $f_1 : {}_fM_f \to B$ one of D-D-bicomodules. Define the components of a morphism $\tilde{f}_1 : {}_f\perp^C(M)_f \to B$ inductively as

$$\tilde{f}_1{}^{(0)} : M^{\otimes^C_0} = C \xrightarrow{f_0} D \xrightarrow{i} B \quad \text{and}$$

$$\tilde{f}_1{}^{(p)} : M^{\otimes^C_p} = M{\otimes^C}M^{\otimes^C_{p-1}} \hookrightarrow M_f{\otimes^D}{}_fM^{\otimes^C_{p-1}} \xrightarrow{f_1 \otimes^D \tilde{f}_1{}^{(p-1)}} B{\otimes^D}B \xrightarrow{m} B \ .$$

Then $\tilde{f} = (\tilde{f}_1, f_0) : \mathcal{T}(\mathcal{G}) \to \mathfrak{D}$ is the desired functor. For instance, the following commutative diagram shows that $\tilde{f}$ preserves compositions.

$$\begin{tikzcd}
M^{\otimes^C_p}{\otimes^C}M^{\otimes^C_q} \arrow[r, hook] \arrow[d, "="'] & M^{\otimes^D_p}{\otimes^D}M^{\otimes^D_q} \arrow[r, "f_1^{\otimes^D_p}{\otimes^D}f_1^{\otimes^D_q}"] \arrow[d, "=" description] & B^{\otimes^D_p}{\otimes^D}B^{\otimes^D_q} \arrow[r, "m_p{\otimes^D}m_q"] \arrow[d, "=" description] & B{\otimes^D}B \ . \arrow[d, "m"] \\
M^{\otimes^C_{p+q}} \arrow[r, hook] & M^{\otimes^D_{p+q}} \arrow[r, "f_1^{\otimes^D_{p+q}}"'] & B^{\otimes^D_{p+q}} \arrow[r, "m_{p+q}"'] & B
\end{tikzcd}$$

(Here $m_1 = i$, $m_2 = m$ and $m_p = m \circ (\mathsf{id}_B{\otimes^D}m_{p-1})$ are the iterated compositions in $\mathfrak{D}$). ☐

Interestingly enough, $\mathcal{T}(\mathcal{G})$ is also free with respect to cofunctors. First notice that, by remark 4.2.1, there is also a canonical cofunctor $\overleftarrow{j} : \mathcal{G} \to \mathcal{T}(\mathcal{G})$ defined by id_C and $M{\otimes^C}C \cong M \xrightarrow{j} \perp^C(M)$. We thus have:

Proposition 9.4.2. *Given any category $\mathfrak{D}$ and a cofunctor (of graphs) $\varphi : \mathcal{G} \to \mathfrak{D}$, there is a unique cofunctor (of categories) $\tilde{\varphi} : \mathcal{T}(\mathcal{G}) \to \mathfrak{D}$ making the following diagram commutative*

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\overleftarrow{j}"] \arrow[dr, "\varphi"'] & \mathcal{T}(\mathcal{G}) \arrow[d, dashed, "\tilde{\varphi}"] \\
 & \mathfrak{D}
\end{tikzcd} .$$

Proof. We are given a morphism of comonoids $\varphi_0 : D \to C$ and one of C-D-bicomodules $\varphi_1 : M{\otimes^C}_{\varphi}D \to {}_{\varphi}B$. This we extend to a morphism $\tilde{\varphi_1} : \perp^C(M){\otimes^C}_{\varphi}D \to {}_{\varphi}\perp^C(M)$ with components

$$\tilde{\varphi_1}^{(0)} : C{\otimes^C}_{\varphi}D \cong D \xrightarrow{i} B \quad \text{and}$$

$$\tilde{\varphi_1}^{(p)} : M^{\otimes^C_p}{\otimes^C}_{\varphi}D = M{\otimes^C}M^{\otimes^C_{p-1}}{\otimes^C}_{\varphi}D \xrightarrow{\mathsf{id}_M \otimes^C \tilde{\varphi_1}^{(p-1)}} M{\otimes^C}_{\varphi}B \cong M{\otimes^C}_{\varphi}D{\otimes^D}B \xrightarrow{\varphi_1 \otimes^D \mathsf{id}_B} B{\otimes^D}B \xrightarrow{m} B \ .$$

Again, it is obvious that $\tilde{\varphi}$ is the desired cofunctor. For instance, preservation of compositions for $\tilde{\varphi}$ is the commutativity of the following diagram, which follows by induction.

$$\begin{tikzcd}
M^{\otimes^C_p}{\otimes^C}M^{\otimes^C_q}{\otimes^C}_{\varphi}D \arrow[r, "\mathsf{id}_p \otimes^C \tilde{\varphi_1}^{(q)}"] \arrow[d, "="'] & M^{\otimes^C_p}{\otimes^C}B \cong M^{\otimes^C_p}{\otimes^C}_{\varphi}D{\otimes^D}B \arrow[r, "\tilde{\varphi_1}^{(p)} \otimes^D \mathsf{id}_B"] & B{\otimes^D}B \arrow[d, "m"] \\
M^{\otimes^C_{p+q}}{\otimes^C}_{\varphi}D \arrow[rr, "\tilde{\varphi_1}^{(p+q)}"'] & & B
\end{tikzcd} .$$

$\square$

A representation of a graph $\mathcal{G} = (M, C)$ is, by definition, a triple (X, p, a) where (X, p) is a left C-comodule and $a : M \otimes^C X \to X$ is an arbitrary morphism of left C-comodules. A morphism of representations is defined as for categories (sectionS:defrepresentations). A representation (X, p, a) of graph $\mathcal{G}$ becomes a representation $(X, p, \tilde{a})$ of the free category $\mathcal{T}(\mathcal{G})$, by extending $a : M \otimes^C X \to X$ to $\tilde{a} : \perp^C(M) \otimes^C X \to X$ so that unitality and associativity hold, as in the above proofs. Conversely, a representation of $\mathcal{T}(\mathcal{G})$ defines a representation of $\mathcal{G}$ by restricting along $\overleftarrow{j}$. These define an isomorphism of categories

$$\mathsf{Rep}_k \mathcal{G} \cong \mathsf{Rep}_k \mathcal{T}(\mathcal{G}) \ .$$

We next discuss ideals and quotients. Let $\mathfrak{C} = (A, C, s, t, i, m)$ be a category in $\boldsymbol{\mathit{Vec}}_k$. A C-C-subbicomodule J of A is called an *ideal* of $\mathfrak{C}$ if

$$m_3(A \otimes^C J \otimes^C A) \subseteq J \ ,$$

where $m_3 = m \circ (m \otimes^C \mathsf{id}_A) = m \circ (\mathsf{id}_A \otimes^C m)$. By unitality this condition is equivalent to

$$m(A \otimes^C J) \subseteq J \text{ and } m(J \otimes^C A) \subseteq J \ .$$

Let M be a C-C-subbicomodule of A. The *ideal generated by M* is

$$J(M) = m_3(A \otimes^C M \otimes^C A) \subseteq A \ .$$

Since m is a morphism of C-C-bicomodules, $J(M)$ is a C-C-subbicomodule. Associativity of m then implies that $J(M)$ is in fact an ideal, and unitality that

$J(M) \supseteq M$. Moreover, it is clear that $J(M)$ is the smallest ideal of $\mathfrak{C}$ containing M.

The *kernel* of a functor $f : \mathfrak{C} \to \mathfrak{D}$ is the space

$$\mathsf{Ker} f = \{a \in A \ / \ f_1(a) = 0\} = \sum \{M \subseteq A \ / \ M \text{ is a } C\text{-}C\text{-subbicomod, } f_1(M) = 0\}.$$

Since f preserves compositions, $\mathsf{Ker} f$ is an ideal of $\mathfrak{C}$. Hence, if $f_1(M) = 0$ then $f_1(J(M)) = 0$.

The *kernel* of a cofunctor $\varphi : \mathfrak{C} \to \mathfrak{D}$ is the space

$$\mathsf{Ker}\varphi = \sum \{M \subseteq A \ / \ M \text{ is a } C\text{-}C\text{-subbicomod and } \varphi_1(M \otimes^C D) = 0 \ \}.$$

Since φ preserves compositions, $\mathsf{Ker}\varphi$ is an ideal of $\mathfrak{C}$. Hence, if $\varphi_1(M \otimes^C D) = 0$ then $\varphi_1(J(M) \otimes^C D) = 0$.

Technical difficulties arise when attempting to define the quotient of a category modulo an ideal, due to the fact that the tensor coproduct of comodules need not preserve epimorphisms. To get around this, one may restrict attention to flat comodules, in view of the results of section A.1. In the applications to be considered in later sections, these assumptions will be satisfied.

Let J be an ideal of a category $\mathfrak{C} = (A, C, s, t, i, m)$. Assume that A and J are flat as right C-comodules, and A/J is flat as left C-comodule. The *quotient* of $\mathfrak{C}$ modulo J is the category $\mathfrak{C}/J = (A/J, C, \bar{s}, \bar{t}, \bar{i}, \bar{m})$, where $(A/J, \bar{s}, \bar{t})$ is the quotient C-C-bicomodule of A modulo J, $\bar{i} : C \xrightarrow{i} A \to A/J$ and

$$\bar{m} : (A/J) \otimes^C (A/J) \cong A \otimes^C A / (J \otimes^C A + A \otimes^C J) \to A/J$$

is the morphism induced by $A \otimes^C A \xrightarrow{m} A \to A/J$. Above we have made use of the canonical isomorphism of proposition A.1.1; this is where the flatness assumptions are needed. The category axioms for $\mathfrak{C}/J$ follow readily from those of $\mathfrak{C}$. In addition, there is a canonical functor $\overrightarrow{\pi} = (\pi, \mathsf{id}_C) : \mathfrak{C} \to \mathfrak{C}/J$, where $\pi : A \to A/J$ is the canonical projection.

To abbreviate, we will say that an ideal J of $\mathfrak{C}$ is *nice* when the above assumptions (on J and A) hold.

Quotient categories satisfy the expected universal property.

Proposition 9.4.3. *Let J be a nice ideal of a category $\mathfrak{C}$, and $f : \mathfrak{C} \to \mathfrak{D}$ a functor such that $f_1(J) = 0$. Then there is a unique functor $\tilde{f} : \mathfrak{C}/J \to \mathfrak{D}$ making the following diagram commutative*

$$\begin{tikzcd}
\mathfrak{C} \arrow[r, "f"] \arrow[d, "\overrightarrow{\pi}"'] & \mathfrak{D} . \\
\mathfrak{C}/J \arrow[ur, dashed, "\tilde{f}"'] &
\end{tikzcd}$$

Proof. Since $f_1(J) = 0$, there is an induced morphism of C-C-bicomodules $\tilde{f}_1 : A/J \to B$. Then $\tilde{f} = (\tilde{f}_1, f_0)$ clearly satisfies the desired property. $\square$

As in the case of free categories, quotient categories are also universal with respect to cofunctors. By remark 4.2.1, there is a canonical cofunctor $\overleftarrow{\pi} : \mathfrak{C} \to \mathfrak{C}/J$ defined by id_C and $A \otimes^C C \cong A \xrightarrow{\pi} A/J$. We thus have:

Proposition 9.4.4. *Let J be a nice ideal of a category $\mathfrak{C}$, and $\varphi : \mathfrak{C} \to \mathfrak{D}$ a cofunctor such that $\varphi_1(J \otimes^C D) = 0$. Assume also that ${}_\varphi D$ is flat as left C-comodule. Then there is a unique cofunctor $\tilde{\varphi} : \mathfrak{C}/J \to \mathfrak{D}$ making the following diagram* *commutative*

$$\begin{tikzcd}
\mathfrak{C} \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathfrak{D} \,. \\
\mathfrak{C}/J \arrow[ur, dashed, "\tilde{\varphi}"'] &
\end{tikzcd}$$

Proof. By assumption, $(A/J)\otimes^{C}D \cong A\otimes^{C}D/J\otimes^{C}D$. Hence the morphism $\varphi_1 : A\otimes^{C}D \to B$ of C-D-bicomodules induces another such $\tilde{\varphi_1} : (A/J)\otimes^{C}D \to B$. Clearly, $\tilde{\varphi} = (\tilde{\varphi_1}, \varphi_0)$ satisfies the desired property. ☐

Let J be a nice ideal of a category $\mathfrak{C}$ and (X, p, a) a representation of $\mathfrak{C}$ such that $a(J\otimes^{C}X) = 0$. If the left C-comodule X is flat, then it becomes a representation of $\mathfrak{C}/J$ via

$$(A/J)\otimes^{C}X \cong A\otimes^{C}X/J\otimes^{C}X \xrightarrow{\tilde{a}} X,$$

where $\tilde{a}$ is the morphism of left C-comodules induced by a.

Now we consider tensor products of quotients. If $J_i \subseteq A_i$ are vector spaces, $i = 1, 2$, then there is a canonical isomorphism

$$(A_1/J_1)\otimes(A_2/J_2) \cong (A_1\otimes A_2)/(J_1\otimes A_2 + A_1\otimes J_2)\ . \qquad (*)$$

Let $\mathfrak{C}_i = (A_i, C_i, \dots)$ be categories in $\boldsymbol{Vec}_k$ and J_i a nice ideal of $\mathfrak{C}_i$, $i = 1, 2$. We claim that then $J_1\otimes A_2 + A_1\otimes J_2$ is a nice ideal of $\mathfrak{C}_1\otimes\mathfrak{C}_2$. In fact, by lemma A.3.5, $A_1\otimes A_2$, $J_1\otimes A_2$, $A_1\otimes J_2$ and $J_1\otimes J_2$ are flat as right $C_1\otimes C_2$-comodules. Since $(J_1\otimes A_2) \cap (A_1\otimes J_2) = J_1\otimes J_2$, lemma A.3.3 applies to conclude that $J_1\otimes A_2 + A_1\otimes J_2$ is flat too. Finally, the isomorphism $(*)$ is one of $C_1\otimes C_2$-$C_1\otimes C_2$-bicomodules, so by lemma A.3.5 $(A_1\otimes A_2)/(J_1\otimes A_2 + A_1\otimes J_2)$ is flat as left $C_1\otimes C_2$-comodule.

Moreover, it is now clear that the pair consisting of the identity of $C_1\otimes C_2$ and the isomorphism $(*)$ yield a canonical isomorphism

$$(\mathfrak{C}_1/J_1)\otimes(\mathfrak{C}_2/J_2) \cong (\mathfrak{C}_1\otimes\mathfrak{C}_2)/(J_1\otimes A_2 + A_1\otimes J_2)\ ,$$

(either in $\overrightarrow{\mathit{Cat}}_{\mathit{Vec}_k}$ or $\overleftarrow{\mathit{Cat}}_{\mathit{Vec}_k}$, by remark 4.2.1).

We are interested in ideals of deltacategories for which the quotient category will inherit the deltacategory structure. It is natural to consider first the case of graphs. A *deltagraph* is a comonoid in the category of graphs and cofunctors, that is a graph $\mathcal{G}$ equipped with coassociative and counital cofunctors (of graphs) $\Delta : \mathcal{G} \to \mathcal{G}\otimes\mathcal{G}$ and $\epsilon : \mathcal{G} \to \mathfrak{I}$. Notice that in this case the base coalgebra C of $\mathcal{G}$ becomes a bialgebra, with multiplication $\Delta_0 : C\otimes C \to C$ and unit $\epsilon_0 : k \to C$.

Let $\mathcal{G} = (M, C, s, t, \Delta, \epsilon)$ be a deltagraph in Vec_k. A C-C-subbicomodule K of M is called a *coideal* of $\mathcal{G}$ if

$$\Delta_1(K\otimes^C(C\otimes C)) \subseteq K\otimes M + M\otimes K \text{ and } \epsilon_1(K\otimes^C k) = 0.$$

A *biideal* of a deltacategory is an ideal of the underlying category that is at the same time a coideal of the underlying deltagraph.

Lemma 9.4.1. *Let K be a coideal of a deltacategory. Then $J(K)$, the ideal generated by K, is a biideal.*

Proof. Let $J = J(K) = m_3(A\otimes^C K\otimes^C A)$. We need to show that J is again a coideal.

First let $\bar{K} = m(A\otimes^C K)$. We have that $\Delta_1(\bar{K}\otimes^C(C\otimes C)) \subseteq J\otimes A + A\otimes J$, as we see from the diagram below (which commutes since Δ preserves compositions):

$$\begin{tikzcd}
A\otimes^C K\otimes^C(C\otimes C) \arrow[r, "\mathsf{id}_A\otimes^C\Delta_1"] \arrow[dd, "m\otimes^C\mathsf{id}"'] & A\otimes^C\left(\begin{smallmatrix} K\otimes A\\ + \\ A\otimes K\end{smallmatrix}\right) \cong A\otimes^C(C\otimes C)\otimes^{C\otimes C}\left(\begin{smallmatrix} K\otimes A\\ + \\ A\otimes K\end{smallmatrix}\right) \arrow[r, "\Delta_1\otimes^{C\otimes C}\mathsf{id}"] & (A\otimes A)\otimes\left(\begin{smallmatrix} K\otimes A\\ + \\ A\otimes K\end{smallmatrix}\right) \arrow[d, "\cong"] \\
 & & \begin{smallmatrix}(A\otimes K)\otimes(A\otimes A)\\ + \\ (A\otimes A)\otimes(A\otimes K)\end{smallmatrix} \arrow[d, "m\otimes m"] \\
\bar{K}\otimes^C(C\otimes C) \arrow[rr, dashed, "\Delta_1"'] & & J\otimes A + A\otimes J
\end{tikzcd}$$

Similarly we now deduce that $\Delta_1(J\otimes^C(C\otimes C)) \subseteq J\otimes A + A\otimes J$, using that $J = m(\bar{K}\otimes^C A)$:

$$\begin{tikzcd}
\bar{K}\otimes^C A\otimes^C(C\otimes C) \arrow[r, "\mathsf{id}_{\bar{K}}\otimes^C\Delta_1"] \arrow[dd, "m\otimes^C\mathsf{id}"'] & \bar{K}\otimes^C(A\otimes A) \cong \bar{K}\otimes^C(C\otimes C)\otimes^{C\otimes C}(A\otimes A) \arrow[r, "\Delta_1\otimes^{C\otimes C}\mathsf{id}"] & (J\otimes A + A\otimes J)\otimes(A\otimes A) \arrow[d, "\cong"] \\
 & & \begin{array}{c}(J\otimes A)\otimes(A\otimes A)+\\ +(A\otimes A)\otimes(J\otimes A)\end{array} \arrow[d, "m\otimes m"] \\
J\otimes^C(C\otimes C) \arrow[rr, dashed, "\Delta_1"'] & & J\otimes A + A\otimes J
\end{tikzcd}$$

The fact that $\epsilon_1(J\otimes^C k) = 0$ is proved in two steps as above, using that ϵ preserves compositions. $\square$

We have now introduced all the terminology required for constructing quotient deltacategories.

Proposition 9.4.5. *Let $(\mathfrak{C}, \Delta, \epsilon)$ be a deltacategory in $\mathtt{Vec}_k$ and J a nice biideal of $\mathfrak{C}$. Assume also that $C\otimes C$ is flat as left C-comodule by corestriction via $\Delta_0 : C\otimes C \to C$. Then the quotient $\mathfrak{C}/J$ inherites a deltacategory structure $(\bar{\Delta}, \bar{\epsilon})$ for which the canonical projection $\overset{\leftharpoonup}{\pi} : \mathfrak{C} \to \mathfrak{C}/J$ is a morphism of deltacategories.*

Proof. By definition of coideal and proposition 9.4.4, $\mathfrak{C} \xrightarrow{\Delta} \mathfrak{C}\otimes\mathfrak{C} \to \mathfrak{C}\otimes\mathfrak{C}/(J\otimes A+A\otimes J)$ and $\mathfrak{C} \xrightarrow{\epsilon} \mathfrak{I}$ factor through J:

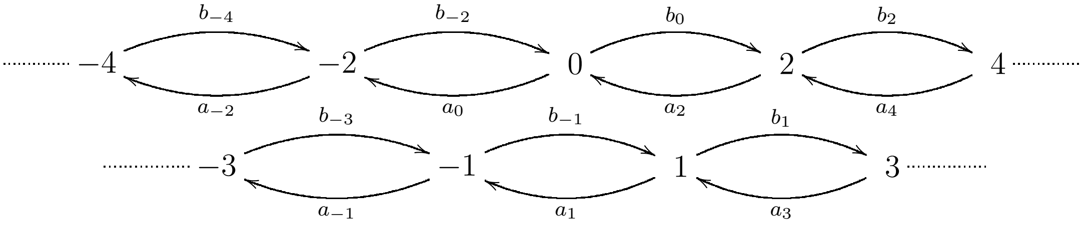
$$\begin{tikzcd}
\mathfrak{C} \arrow[r, "\Delta"] \arrow[d, "\overleftarrow{\pi}"'] & \mathfrak{C}\otimes\mathfrak{C} \arrow[d, "\overleftarrow{\pi}\otimes\overleftarrow{\pi}"'] \arrow[dr] & \\
\mathfrak{C}/J \arrow[r, dashed] & \mathfrak{C}/J\otimes\mathfrak{C}/J \arrow[r, "\cong"'] & \mathfrak{C}\otimes\mathfrak{C}/(J\otimes A + A\otimes J)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{C} \arrow[r, "\epsilon"] \arrow[d, "\overleftarrow{\pi}"'] & \mathfrak{I}\,. \\
\mathfrak{C}/J \arrow[ur, dashed, "\bar{\epsilon}"'] &
\end{tikzcd}$$

We let $\bar{\Delta} : \mathfrak{C}/J \to \mathfrak{C}/J\otimes\mathfrak{C}/J$ be the composite of the bottom row in the first diagram, and $\bar{\epsilon} : \mathfrak{C} \to \mathfrak{C}/J$ be as shown. Coassociativity and counitality for $(\mathfrak{C}/J, \bar{\Delta}, \bar{\epsilon})$ follow from those for $(\mathfrak{C}, \Delta, \epsilon)$, plus uniqueness of quotient factorizations. By construction, $\overleftarrow{\pi} : \mathfrak{C} \to \mathfrak{C}/J$ preserves the deltacategory structures. $\square$

Remark 9.4.1. Let $(\mathfrak{C}, \Delta, \epsilon)$ be a deltacategory in $\boldsymbol{\mathit{Vec}}_k$, with base coalgebra C. Then $(C, \Delta_0, \epsilon_0)$ is a bialgebra. If in addition C happens to be a Hopf algebra, then the hypothesis in proposition 9.4.5 that $C\otimes C$ be flat as left C-comodule is automatically satisfied, according to examples A.1.1.

9.5 $U_q(sl_2)$ as admissible sections

Consider the following (small) graph $\mathcal{G}$ in $\boldsymbol{\mathit{Sets}}$



The set of objects is $\mathbb{Z}$, and for each object $i \in \mathbb{Z}$ there are two arrows with source i, b_i and a_i, with targets $i+2$ and $i-2$ respectively.

Applying the *linearization* functor $\boldsymbol{\mathit{Sets}} \to \boldsymbol{\mathit{Vec}}_k$, $S \mapsto kS$, we obtain a k-linear graph $k\mathcal{G} = (M, C)$ with $C = k\mathbb{Z} = k[c, c^{-1}]$ and $M = \oplus_{i \in \mathbb{Z}} ka_i \oplus kb_i$. We are viewing $k\mathcal{G}$ as a graph in $\boldsymbol{\mathit{Vec}}_k$ as explained in section 9.1. Consider the free category $\mathcal{T}(k\mathcal{G})$ on this graph (section 9.4), $\mathcal{T}(k\mathcal{G}) = (A, C, \dots)$. Let $e_i \in A$ denote the identity of i in $\mathcal{T}(k\mathcal{G})$.

Fix an arbitrary scalar $q \in k$. We claim that $\mathcal{T}(k\mathcal{G})$ is a deltacategory. We first define cofunctors of graphs $\Delta : k\mathcal{G} \to \mathcal{T}(k\mathcal{G}) {\otimes} \mathcal{T}(k\mathcal{G})$ and $\epsilon : k\mathcal{G} \to \mathfrak{I}$ as follows:

$$\begin{array}{rcl} \Delta_0 : C{\otimes}C & \to & C \\ c^i{\otimes}c^j & \mapsto & c^{i+j} \end{array} \quad \text{and} \quad \begin{array}{rcl} \Delta_1 : M{\otimes^C}(C{\otimes}C) & \to & A{\otimes}A \\ b_{i+j}{\otimes}(c^i{\otimes}c^j) & \mapsto & e_i{\otimes}b_j + q^j b_i{\otimes}e_j \\ a_{i+j}{\otimes}(c^i{\otimes}c^j) & \mapsto & a_i{\otimes}e_j + q^{-i} e_i{\otimes}a_j \end{array}$$

and

$$\begin{array}{rcl} \epsilon_0 : k & \to & C \\ 1 & \mapsto & c^0 \end{array} \quad \text{and} \quad \begin{array}{rcl} \epsilon_1 : M{\otimes^C}k & \to & k \\ b_0 & \mapsto & 0 \\ a_0 & \mapsto & 0 \end{array} \cdot$$

Clearly, Δ and ϵ are cofunctors of graphs (for instance, the targets of $e_i{\otimes}b_j$ and $q^j b_i{\otimes}e_j$ are $(i, j+2)$ and $(i+2, j)$, which map by Δ_0 to $i+j+2$, the target of b_{i+j}). By proposition 9.4.2, they extend to cofunctors (of categories) $\Delta : \mathcal{T}(k\mathcal{G}) \to \mathcal{T}(k\mathcal{G}){\otimes}\mathcal{T}(k\mathcal{G})$ and $\epsilon : \mathcal{T}(k\mathcal{G}) \to \mathfrak{I}$. By the uniqueness in proposition 9.4.2, it is enough to check coassociativity and counitality for Δ and ϵ on the generating graph $k\mathcal{G}$.

In order to do this, notice that we have

$$\Delta_1(e_{i+j}{\otimes}(c^i{\otimes}c^j)) = e_i{\otimes}e_j \quad \text{and} \quad \epsilon_1(e_0) = 1 \ ,$$

since by construction Δ and ϵ preserve identities.

Now, coassociativity for Δ boils down (by definition of composition of cofunctors in section 4.2) to associativity for the multiplication Δ_0, which is clear, plus equality between the following two maps $M\otimes^{C}(C\otimes C\otimes C) \to A\otimes A\otimes A$

$$((\Delta\otimes \mathsf{id}) \circ \Delta)_1 \text{ and } ((\mathsf{id}\otimes\Delta) \circ \Delta)_1 \ .$$

A k-basis for $M\otimes^{C}(C\otimes C\otimes C)$ is $\{b_{i+j+k}\otimes(c^i\otimes c^j\otimes c^k),\ a_{i+j+k}\otimes(c^i\otimes c^j\otimes c^k) \ / \ i,j,k \in \mathbb{Z}\}$. We will check that those maps agree on the first of these elements, the other case is similar. According to the description of composition of cofunctors in section 9.1, the element $((\Delta\otimes\mathsf{id}) \circ \Delta)_1(b_{i+j+k}\otimes(c^i\otimes c^j\otimes c^k))$ is called the lift of b_{i+j+k} to (i,j,k) by $(\Delta\otimes\mathsf{id}) \circ \Delta$, and it is computed by first lifting b_{i+j+k} to $(i+j,k)$ by Δ and then lifting the result to (i,j,k) by $\Delta\otimes\mathsf{id}$, as done below

$$\begin{tikzcd}
\mathcal{T}\otimes\mathcal{T}\otimes\mathcal{T} & (i,j,k) \arrow[d, mapsto] & e_i\otimes e_j\otimes b_k + q^k(e_i\otimes b_j + q^j b_i\otimes e_j)\otimes e_k \ . \\
\mathcal{T}\otimes\mathcal{T} \arrow[u, "\Delta\otimes\mathsf{id}"] & (i+j,k) \arrow[d, mapsto] & e_{i+j}\otimes b_k + q^k b_{i+j}\otimes e_k \arrow[u, mapsto] \\
\mathcal{T} \arrow[u, "\Delta"] & i+j+k & b_{i+j+k} \arrow[u, mapsto]
\end{tikzcd}$$

Similarly, the lift of b_{i+j+k} to (i,j,k) by $(\mathsf{id}\otimes\Delta) \circ \Delta$, that is, the element $((\mathsf{id}\otimes\Delta) \circ \Delta)_1(b_{i+j+k}\otimes(c^i\otimes c^j\otimes c^k))$, is

$$\begin{tikzcd}
\mathcal{T}\otimes\mathcal{T}\otimes\mathcal{T} & (i,j,k) \arrow[d, mapsto] & e_i\otimes(e_j\otimes b_k + q^k b_j\otimes e_k) + q^{j+k} b_i\otimes e_j\otimes e_k \ . \\
\mathcal{T}\otimes\mathcal{T} \arrow[u, "\mathsf{id}\otimes\Delta"] & (i,j+k) \arrow[d, mapsto] & e_i\otimes b_{j+k} + q^{j+k} b_i\otimes e_{j+k} \arrow[u, mapsto] \\
\mathcal{T} \arrow[u, "\Delta"] & i+j+k & b_{i+j+k} \arrow[u, mapsto]
\end{tikzcd}$$

Thus, the two lifts give the same element $e_i{\otimes}e_j{\otimes}b_k + q^k e_i{\otimes}b_j{\otimes}e_k + q^{j+k} b_i{\otimes}e_j{\otimes}e_k$, as needed. Counitality is checked similarly. This completes the proof of the claim that $(\mathcal{T}(k\mathcal{G}), \Delta, \epsilon)$ is a deltacategory. We further claim that, if $q^2 \neq 1$, the relations

$$b_{i-2} \circ a_i - a_{i+2} \circ b_i = (i)_q e_i, \quad \text{for each} \quad i \in \mathbb{Z}\ , \tag{*}$$

where $(i)_q = \dfrac{q^i - q^{-i}}{q - q^{-1}}$, define a nice biideal of $\mathcal{T}(k\mathcal{G})$. More precisely, consider the k-subspace R of A linearly spanned by the elements $r_i := b_{i-2} \circ a_i - a_{i+2} \circ b_i - (i)_q e_i$ for $i \in \mathbb{Z}$. Since each r_i is a linear combination of arrows with the same source and target i, R is a C-C-subbicomodule of A. Hence, the ideal $J(R)$ generated by R is defined (section 9.4). It is a nice ideal because, over a group-like coalgebra like $C = k\mathbb{Z}$, every comodule is flat (section A.1). To check that $J(R)$ is a biideal of $\mathcal{T}(k\mathcal{G})$, it suffices to show that R is a coideal, by lemma 9.4.1. To see this, we need to compute the lift of r_{i+j} to (i,j). The lift of $b_{i+j-2} \circ a_{i+j}$ is computed as follows. We first compute the lift of a_{i+j} to (i,j). This involves various arrows with different targets. We then compute the lifts of b_{i+j-2} to these targets, as below:

$$\begin{tikzcd}
(i,j) \arrow[d, mapsto] & \begin{array}{c} a_i{\otimes}e_j \\ +q^{-i}e_i{\otimes}a_j \end{array} & \begin{array}{c}(i-2,j)\\(i,j-2)\end{array} \arrow[d, mapsto] & \begin{array}{c} e_{i-2}{\otimes}b_j + q^j b_{i-2}{\otimes}e_j \\ e_i{\otimes}b_{j-2} + q^{j-2} b_i{\otimes}e_{j-2} \end{array} . \\
i+j & a_{i+j} \arrow[u, mapsto] & i+j-2 & b_{i+j-2} \arrow[u, mapsto]
\end{tikzcd}$$

Since Δ preserves compositions, composing the top rows we find the lift of $b_{i+j-2} \circ a_{i+j}$ to (i,j); it is

$$a_i{\otimes}b_j + q^j (b_{i-2} \circ a_i){\otimes}e_j + q^{-i} e_i{\otimes}(b_{j-2} \circ a_j) + q^{j-i-2} b_i{\otimes}a_j\ .$$

Similarly, the lifts of $a_{i+j+2} \circ b_{i+j}$ and $(i+j)_q e_{i+j}$ to (i,j) are respectively

$$a_i{\otimes}b_j + q^{-i} e_i{\otimes}(a_{j+2} \circ b_j) + q^j (a_{i+2} \circ b_i){\otimes}e_j + q^{j-i-2} b_i{\otimes}a_j \text{ and } (i+j)_q e_i{\otimes}e_j\ .$$

Hence the lift of r_{i+j} to (i,j) is

$$\begin{aligned} &a_i\otimes b_j + q^j(b_{i-2}\circ a_i)\otimes e_j + q^{-i}e_i\otimes(b_{j-2}\circ a_j) + q^{j-i-2}b_i\otimes a_j \\ &- a_i\otimes b_j - q^{-i}e_i\otimes(a_{j+2}\circ b_j) - q^j(a_{i+2}\circ b_i)\otimes e_j - q^{j-i-2}b_i\otimes a_j \\ &- (i+j)_q e_i\otimes e_j \end{aligned}$$

Using the well-known identity $(i+j)_q = q^j(i)_q + q^{-i}(j)_q$, this element becomes

$$\begin{aligned} &= q^j\Big(b_{i-2}\circ a_i - a_{i+2}\circ b_i - (i)_q e_i\Big)\otimes e_j + q^{-i}e_i\otimes\Big(b_{j-2}\circ a_j - a_{j+2}\circ b_j - (j)_q e_i\Big) \\ &= q^j r_i\otimes e_j + q^{-i}e_i\otimes r_j \ \in \ R\otimes A + A\otimes R\ . \end{aligned}$$

This proves that $\Delta_1(R\otimes^C(C\otimes C)) \subseteq R\otimes A + A\otimes R$. Similarly one shows that $\epsilon_1(R\otimes^C k) = 0$. This completes the proof of the claim that $J(R)$ is a nice biideal.

It follows now from proposition 9.4.5 that the quotient $\mathfrak{C} = \mathcal{T}(k\mathcal{G})/J(R)$ carries a structure of deltacategory.

Now consider the following admissible sections of $\mathfrak{C}$:

$$\begin{array}{rclcrclcrcl} K : C & \to & A, & & E : C & \to & A & \text{and} & F : C & \to & A \\ c^i & \mapsto & q^i e_i & & c^i & \mapsto & b_i & & c^i & \mapsto & a_i \end{array}$$

One checks immediately that

(1) K is invertible, with $K^{-1}(c^i) = q^{-i}e_i$,

(2) $KE = q^2EK$ and $KF = q^{-2}FK$, and

(3) $EF - FE = \dfrac{K-K^{-1}}{q-q^{-1}}$.

For instance:

$$(K*E)(c^i) = m(K\otimes^C\mathsf{id})tE(c^i) = m(K\otimes^C\mathsf{id})(c^{i+2}\otimes b_i) = q^{i+2}m(e_{i+2}\otimes b_i) = q^{i+2}b_i$$

while

$$(E * K)(c^i) = m(E \otimes^{C} \mathsf{id}) tK(c^i) = m(E \otimes^{C} \mathsf{id})(q^i c^i \otimes e_i) = q^i m(b_i \otimes e_i) = q^i b_i$$

from where $KE = q^2 EK$. Similarly for the other half of (2). Also,

$$(E * F)(c^i) = m(E \otimes^{C} \mathsf{id}) tF(c^i) = m(E \otimes^{C} \mathsf{id})(c^{i-2} \otimes a_i) = m(b_{i-2} \otimes a_i) = b_{i-2} \circ a_i$$

and

$$(F * E)(c^i) = m(F \otimes^{C} \mathsf{id}) tE(c^i) = m(F \otimes^{C} \mathsf{id})(c^{i+2} \otimes b_i) = m(a_{i+2} \otimes b_i) = a_{i+2} \circ b_i,$$

hence

$$(EF - FE)(c^i) = b_{i-2} \circ a_i - a_{i+2} \circ b_i \overset{(*)}{=} (i)_q e_i = \frac{q^i - q^{-i}}{q - q^{-1}} e_i = \frac{K - K^{-1}}{q - q^{-1}} (c^i),$$

which proves (3).

Let $\Gamma_0(\mathfrak{C})$ denote the k-subalgebra of $\Gamma(\mathfrak{C})$ generated by K, E and F. The above shows that there is an epimorphism of k-algebras

$$U_q(sl_2) \twoheadrightarrow \Gamma_0(\mathfrak{C}) ,$$

since $U_q(sl_2)$ is defined by generators K, E and F subject precisely to relations (1)-(3) [K, VI.1.1]. We claim that, if q is not a root of unity, then this map is an isomorphism. We will return to the proof of this claim at the end of the section.

Even though $\mathfrak{C}$ is not finite, it turns out that $\Gamma(\Delta) : \Gamma_0(\mathfrak{C}) \to \Gamma(\mathfrak{C} \otimes \mathfrak{C})$ has its image in the subalgebra $\Gamma_0(\mathfrak{C}) \otimes \Gamma_0(\mathfrak{C})$ of $\Gamma(\mathfrak{C} \otimes \mathfrak{C})$. For instance, let us find $\Gamma(\Delta)(E)$, following the definition in section 5.3:

$$\begin{array}{ccccccccc}
C \otimes C & \xrightarrow{\Delta_{C \otimes C}} & (C \otimes C) \otimes^{C} (C \otimes C) & \xrightarrow{\Delta_0 \otimes^{C} \mathsf{id}} & C \otimes^{C} (C \otimes C) & \xrightarrow{E \otimes^{C} \mathsf{id}} & A \otimes^{C} (C \otimes C) & \xrightarrow{\Delta_1} & A \otimes A \ ; \\
c^i \otimes c^j & \mapsto & c^i \otimes c^j \otimes c^i \otimes c^j & \mapsto & c^{i+j} \otimes c^i \otimes c^j & \mapsto & b_{i+j} \otimes c^i \otimes c^j & \mapsto & \begin{smallmatrix} e_i \otimes b_j \\ + \\ q^j b_i \otimes e_j \end{smallmatrix}
\end{array}$$

thus, $\Gamma(\Delta)(E)(c^i \otimes c^j) = e_i \otimes b_j + q^j b_i \otimes e_j = (1 \otimes E + E \otimes K)(c^i \otimes c^j)$; hence $\Gamma(\Delta)(E) = 1 \otimes E + E \otimes K \in \Gamma_0(\mathfrak{C}) \otimes \Gamma_0(\mathfrak{C})$. Also,

$$\begin{array}{rccccccccc} \Gamma(\epsilon)(E) : I & \xrightarrow{\Delta_I} & I \otimes^C I & \xrightarrow{\epsilon_0 \otimes^C \mathsf{id}_I} & C \otimes^C I & \xrightarrow{E \otimes^C \mathsf{id}_I} & A \otimes^C I & \xrightarrow{\epsilon_1} & I \\ 1 & \mapsto & 1 \otimes 1 & \mapsto & c^0 \otimes 1 & \mapsto & b_0 \otimes 1 & \mapsto & 0 \end{array}$$

hence $\Gamma(\epsilon)(E) = 0$. Similarly,

$$\begin{array}{ll} \Gamma(\Delta)(F) = K^{-1} \otimes F + F \otimes 1 & \qquad \Gamma(\epsilon)(F) = 0 \\ \Gamma(\Delta)(K) = K \otimes K & \qquad \Gamma(\epsilon)(K) = 1 \end{array}$$

which proves that the map $U_q(sl_2) \twoheadrightarrow \Gamma_0(\mathfrak{C})$ is a morphism of k-coalgebras too [K, VII.1.1].

Before proving that this map is actually an isomorphism, we define an action of $\Gamma(\mathfrak{C})$ (and hence also of $\Gamma_0(\mathfrak{C})$) on the *quantum plane* $k_q[x, y] = k\langle x, y\rangle/(xy = qyx)$ [K, IV.1]. By corollary 9.2.1, it is enough to show that $k_q[x, y]$ is a representation of $\mathfrak{C}$; in turn, in view of the remarks about representations of free and quotient categories in section 9.4, it is enough to define a representation of the graph $\mathcal{G}$ for which relations $(*)$ are preserved. This is as follows

$$\begin{array}{rclcrcl} p : k_q[x, y] & \to & C \otimes k_q[x, y] & \qquad & a : M \otimes^C k_q[x, y] & \to & k_q[x, y] \\ x^m y^n & \mapsto & c^{m-n} \otimes x^m y^n & & b_{m-n} \otimes^C x^m y^n & \mapsto & (n)_q x^{m+1} y^{n-1} \\ & & & & a_{m-n} \otimes^C x^m y^n & \mapsto & (m)_q x^{m-1} y^{n+1} \end{array}$$

The fact that relations $(*)$ are preserved by a boils down to the well-known identity

$$(n+1)_q (m)_q - (n)_q (m+1)_q = (m-n)_q \ .$$

The resulting action of $\Gamma_0(\mathfrak{C})$ on $k_q[x,y]$ is:

$$\begin{aligned} K\cdot x^m y^n &= q^{m-n}x^m y^n \\ E\cdot x^m y^n &= (n)_q x^{m+1}y^{n-1} \\ F\cdot x^m y^n &= (m)_q x^{m-1}y^{n+1} \end{aligned}$$

For instance the action of E is, according to remark 9.2.1,

$$\begin{array}{ccccccc} k_q[x,y]: & \xrightarrow{p} & C{\otimes^C}k_q[x,y] & \xrightarrow{E\otimes^C\mathsf{id}} & A{\otimes^C}k_q[x,y] & \xrightarrow{a} & k_q[x,y] \\ x^m y^n & \mapsto & c^{m-n}{\otimes}x^m y^n & \mapsto & b_{m-n}{\otimes}x^m y^n & \mapsto & (n)_q x^{m+1}y^{n-1} \end{array}$$

as claimed.

This means that there is a commutative diagram

$$\begin{tikzcd} U_q(sl_2) \arrow[r, two heads] \arrow[dr, hook] & \Gamma_0(\mathfrak{C}) \arrow[d] \\ & \mathsf{End}_k(k_q[x,y]) \end{tikzcd} ,$$

where $U_q(sl_2)\to\mathsf{End}_k(k_q[x,y])$ is the canonical action on the quantum plane [K, VII.3.3]. Now, it is well-known that this map is injective when q is not a root of unity. (We provide a proof for completeness. By, theorems VII.2.2, VII.3.3.b and VI.3.5 in [K], if $u\in U_q(sl_2)$ is in the kernel of this map, then it annihilates any finite-dimensional $U_q(sl_2)$-module *of type 1* [Jan, 5.2]. But then $u=0$ by [Jan, 5.11 and 5.4] .) It follows from the commutativity of the diagram that the map

$$U_q(sl_2)\to\Gamma_0(\mathfrak{C})$$

is an isomorphism.

The above construction of $U_q(sl_2)$ resembles that of Cibils and Rosso [CR]. There $U_q(sl_2)$ is obtained as a quotient of the path algebra of the graph we started with;

here we introduce all the relevant structure at the more basic level of the category (the relations and the comonoid structure) and then pass to the associated algebra. We will proceed in the same way for Drinfeld's double below; here consideration of general internal categories in $\mathtt{Vec}_k$ is essential, for the category in question is not linear.

9.6 Drinfeld's double as admissible sections

Let H be a finite dimensional Hopf algebra with antipode λ. Drinfeld's double is $D(H) = H \bowtie (H^*)^{op}$ with multiplication

$$(a \bowtie f) \cdot (b \bowtie g) = f_1(b_1)(ab_2 \bowtie g f_2) f_3(\lambda b_3)$$

where the product of g and f_2 and the diagonalization of f are both in the bialgebra H^*. (This form of Drinfeld's double is sometimes called the right handed version [Maj, 7,1,1], to distinguish it from the more common left handed version of [K, IX.4.1] and [Maj, 7.1.2]).

We define a category $\mathfrak{D}_H = (H{\otimes}H, H, s, t, i, m)$ in $\mathtt{Vec}_k$ as follows:

$$\begin{array}{rclrcl}
s : H{\otimes}H & \to & (H{\otimes}H){\otimes}H, & a{\otimes}b & \mapsto & a{\otimes}b_1{\otimes}b_2 \\
t : H{\otimes}H & \to & H{\otimes}(H{\otimes}H), & a{\otimes}b & \mapsto & a_1 b_1 \lambda(a_3){\otimes}a_2{\otimes}b_2 \\
i : H & \to & H{\otimes}H, & a & \mapsto & 1{\otimes}a \\
m : (H{\otimes}H)\overset{H}{\otimes}(H{\otimes}H) & \to & H{\otimes}H, & a{\otimes}b{\otimes}c{\otimes}d & \mapsto & \epsilon(b)ac{\otimes}d
\end{array}$$

The category axioms (definition 2.3.1) are easily checked. Notice that $\mathfrak{D}_H$ is defined for any Hopf algebra H. We claim that if H is finite dimensional, so that $D(H)$ is defined, then $\Gamma(\mathfrak{D}_H) = D(H)$.

To see this, recall from remark 3.0.1 that if V is a k-space, $V \otimes C$ is viewed as right C-comodule via $\mathsf{id}_V \otimes \Delta_C$, and (U, p) is a right C-comodule, then

$$\mathsf{Hom}_C(U, V \otimes C) \cong \mathsf{Hom}_k(U, V)$$

under the maps

$$u : U \to V \otimes C \mapsto \tilde{u} : U \xrightarrow{u} V \otimes C \xrightarrow{\mathsf{id}_V \otimes \epsilon_C} V \text{ and } \tilde{u} : U \to V \mapsto U \xrightarrow{p} U \otimes C \xrightarrow{\tilde{u} \otimes \mathsf{id}_C} V \otimes C$$

It follows from this and the definition of s that, as k-spaces,

$$\Gamma(\mathfrak{D}_H) = \mathsf{Hom}_H(H, H \otimes H) \cong \mathsf{Hom}_k(H, H) \cong H \otimes H^* = D(H) \ .$$

We need to check that this is an isomorphism of k-algebras. Let u and v in $\Gamma(\mathfrak{D}_H)$ be the images of $a \bowtie f$ and $b \bowtie g$ in $D(H)$. Thus, $\forall \ h \in H$,

$$u(h) = \tilde{u}(h_1) \otimes h_2 = f(h_1) a \otimes h_2 \text{ and } v(h) = \tilde{v}(h_1) \otimes h_2 = g(h_1) b \otimes h_2 \ .$$

We compute:

$$(u * v)(h) = m(u \otimes^H \mathsf{id}) t v(h) = m(u \otimes^H \mathsf{id}) t(g(h_1) b \otimes h_2) = m(u \otimes^H \mathsf{id})(g(h_1) b_1 h_2 \lambda b_3 \otimes b_2 \otimes h_3)$$

$$= m(g(h_1) f(b_1 h_2 \lambda b_5) a \otimes b_2 h_3 \lambda b_4 \otimes b_3 \otimes h_4) = g(h_1) f(b_1 h_2 \lambda b_5) \epsilon(b_2 h_3 \lambda b_4) a b_3 \otimes h_4$$

$$= g(h_1) f(b_1 h_2 \lambda b_3) a b_2 \otimes h_3 \ .$$

On the other hand, let $w \in \Gamma(\mathfrak{D}_H)$ be the image of

$$(a \bowtie f) \cdot (b \bowtie g) = f_1(b_1) f_3(\lambda b_3)(a b_2 \bowtie g f_2) \ .$$

Then,

$$w(h) = f_1(b_1)f_3(\lambda b_3)(gf_2)(h_1)ab_2 \otimes h_2 = f_1(b_1)f_3(\lambda b_3)g(h_1)f_2(h_2)ab_2 \otimes h_3 =$$

$$= g(h_1)f(b_1h_2\lambda b_3)ab_2 \otimes h_3 = (u * v)(h);$$

thus, $(u * v) = w$.

Also, $1 \bowtie \epsilon \in D(H)$ clearly maps to $i \in \Gamma(\mathfrak{D}_H)$. Thus $\Gamma(\mathfrak{D}_H) \cong D(H)$ as k-algebras.

The coalgebra structure on $D(H)$ (the usual structure on the tensor product of the coalgebras H and H^*) can also be recovered from the category; one easily checks that it comes from the following deltacategory structure on $\mathfrak{D}_H$: $\Delta : \mathfrak{D}_H \to \mathfrak{D}_H \otimes \mathfrak{D}_H$ is the cofunctor $\Delta = (\Delta_1, \Delta_0)$ given by $\Delta_0 : H \otimes H \xrightarrow{\mu_H} H$ and

$$\Delta_1 : (H \otimes H) \otimes^H (H \otimes H) \xrightarrow[\cong]{\mathrm{id} \otimes \epsilon_H \otimes \mathrm{id} \otimes \mathrm{id}} H \otimes (H \otimes H) \xrightarrow{\Delta_H \otimes \mathrm{id}} (H \otimes H) \otimes (H \otimes H)$$

and $\epsilon : \mathfrak{D}_H \to \mathfrak{I}$ is the cofunctor $\epsilon = (\epsilon_1, \epsilon_0)$ given by

$$\epsilon_0 : k \xrightarrow{\mathrm{u}_H} H \text{ and } \epsilon_1 : (H \otimes H) \otimes^H k \xrightarrow[\cong]{\mathrm{id} \otimes \epsilon_H} H \xrightarrow{\epsilon_H} k \ .$$

The description of $D(H)$ as admissible sections of the category $\mathfrak{D}_H$ can be used to derive many of its properties, some familiar, some new. We list some next.

1. Subalgebras of the double. A morphism $f : H \to K$ of Hopf algebras induces a functor $(f \otimes f, f) : \mathfrak{D}_H \to \mathfrak{D}_K$; hence also, by corollary 5.4.1, an order-preserving correspondence

$$f^{-1} : \wp(D(K)) \to \wp(D(H))$$

where $\wp(D(H))$ denotes either the lattice of subsets, subspaces or subalgebras of $D(H)$. This result seems to be new.

2. <u>The square of the antipode.</u> Consider the functors $id = (\mathsf{id}_{H\otimes H}, \mathsf{id}_H) : \mathfrak{D}_H \to \mathfrak{D}_H$ and $g = (\lambda^2_H \otimes \lambda^2_H, \lambda^2_H) : \mathfrak{D}_H \to \mathfrak{D}_H$. When H is finite dimensional, both id_H and λ^2_H are isomorphisms, so id and g can be seen as cofunctors $\mathfrak{D}_H \to \mathfrak{D}_H$ (remark 4.2.1). The induced morphisms $D(H) \to D(H)$ are the identity and the square of the antipode of $D(H)$ respectively.

   Now, one checks easily that there is a natural isomorphism $\alpha : id \Rightarrow g$ defined by $\alpha : H \to H \otimes H$, $\alpha(h) = \lambda h_1 \otimes h_2$. This can be seen as a natural coisomorphism between the corresponding cofunctors. This implies, by proposition 5.3.2, that the square of the antipode of $D(H)$ is given by conjugation by $\alpha \in \Gamma(\mathfrak{D}_H) = D(H)$. One can then obtain the same conclusion for any finite dimensional quasitriangular Hopf algebra in place of $D(H)$, since any such is a quotient of its double. (A direct proof of this result can be found in [K, VIII.4.1]).

3. <u>Yetter-Drinfeld modules.</u> The well-known description of (left) $D(H)$-modules in terms of (left) *Yetter-Drinfeld modules* [Mon, 10.6.16] is an immediate consequence of the description of $D(H)$ as admissible sections, plus proposition 9.2.4 (notice that $H \otimes H$ is free as right H-comodule, in particular flat, so this proposition applies). In fact, writing down the definition of left $\mathfrak{D}_H$-representations one finds that it becomes precisely that of left Yetter-Drinfeld modules. Let us provide the details of this claim.

A left Yetter-Drinfeld H-module (sometimes called a crossed H-bimodule) is a k-space X equipped with a left H-module structure $\chi : H \otimes X \to X$ and a left H-comodule structure $p : X \to H \otimes X$, such that

$$\begin{tikzcd}
 & H\otimes H\otimes H\otimes X \arrow[rrrr, "\mathsf{id}_H\otimes\tau_{H,H}\otimes\mathsf{id}_X"] & & & & H\otimes H\otimes H\otimes X \arrow[dr, "\mu_H\otimes\chi"] & \\
H\otimes X \arrow[ur, "\Delta_H\otimes p"] \arrow[d, "\Delta_H\otimes\mathsf{id}_X"'] & & & & & & H\otimes X \\
H\otimes H\otimes X \arrow[dr, "\mathsf{id}_H\otimes\tau_{H,X}"'] & & & & & & H\otimes H\otimes X \arrow[u, "\mu_H\otimes\mathsf{id}_X"'] \\
 & H\otimes X\otimes H \arrow[rr, "\chi\otimes\mathsf{id}_H"'] & & X\otimes H \arrow[rr, "p\otimes\mathsf{id}_H"'] & & H\otimes X\otimes H \arrow[ur, "\mathsf{id}_H\otimes\tau_{X,H}"'] &
\end{tikzcd}$$

commutes. Writing $\chi(h\otimes x) = h\cdot x$ and $p(x) = x_{-1}\otimes x_0$, this condition becomes

$$(h_1\cdot x)_{-1}h_2\otimes(h_1\cdot x)_0 = h_1x_{-1}\otimes h_2\cdot x_0\ . \tag{YD}$$

Now, by definition 6.1.1, a representation of $\mathfrak{D}_H$ is a k-space X equipped with a left H-comodule structure $p : X \to H\otimes X$ and a morphism of left H-comodules $a : (H\otimes H)\otimes^H X \to X$ such that diagrams below commute

$$\begin{tikzcd}
(H\otimes H)\otimes^H X \arrow[r, "a"] \arrow[dr, phantom, "(1)"] & X \arrow[dl, bend left, "\cong", "p"'] \\
H\otimes^H X \arrow[u, "i\otimes^H\mathsf{id}_X"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(H\otimes H)\otimes^H(H\otimes H)\otimes^H X \arrow[r, "\mathsf{id}_{H\otimes H}\otimes^H a"] \arrow[d, "m\otimes^H\mathsf{id}_X"'] \arrow[dr, phantom, "(2)"] & (H\otimes H)\otimes^H X \arrow[d, "a"] \\
(H\otimes H)\otimes^H X \arrow[r, "a"'] & X
\end{tikzcd}\ .$$

By propositions 2.2.2 and 2.2.3 and the definition of s, $H\otimes X \xrightarrow{\mathsf{id}_H\otimes p} (H\otimes H)\otimes^H X$ is an isomorphism, with inverse $(H\otimes H)\otimes^H X \xrightarrow{\mathsf{id}_H\otimes\epsilon\otimes\mathsf{id}_X} H\otimes X$. Let $\chi = a(\mathsf{id}_H\otimes p)$: $H\otimes X \to X$, so that $a = \chi(\mathsf{id}_H\otimes\epsilon\otimes\mathsf{id}_X) : (H\otimes H)\otimes^H X \to X$.

Let us reformulate the conditions on a in terms of χ, writing $p(x) = x_{-1}\otimes x_0$

and $\chi(h \otimes x) = h \cdot x$. First,

$$(1) \Leftrightarrow a(1 \otimes x_{-1} \otimes x_0) = x \Leftrightarrow \chi\Big(\epsilon(x_{-1})1 \otimes x_0\Big) = x \Leftrightarrow 1 \cdot x = x \ \forall \ x \in X \ .$$

As for (2) notice that, by the same reasons above, there is an isomorphism

$$H \otimes H \otimes X \xrightarrow{\cong} (H \otimes H) \otimes^{H} (H \otimes H) \otimes^{H} X, \quad h \otimes k \otimes x \mapsto h \otimes k_1 x_{-2} \lambda(k_3) \otimes k_2 \otimes x_{-1} \otimes x_0 \ .$$

Therefore, (2) holds if and only if, $\forall \ h \otimes k \otimes x \in H \otimes H \otimes X$,

$$\begin{aligned} a(m \otimes^{H} \mathsf{id}_X)(h \otimes k_1 x_{-2} \lambda(k_3) \otimes k_2 \otimes x_{-1} \otimes x_0) &= a(\mathsf{id}_{H \otimes H} \otimes^{H} a)(h \otimes k_1 x_{-2} \lambda(k_3) \otimes k_2 \otimes x_{-1} \otimes x_0) \\ \Leftrightarrow a\Big(\epsilon(k_1 x_{-2} \lambda(k_3)) h k_2 \otimes x_{-1} \otimes x_0\Big) &= a\Big(h \otimes k_1 x_{-2} \lambda k_3 \otimes \chi(\epsilon(x_{-1}) k_2 \otimes x_0)\Big) \\ \Leftrightarrow a(hk \otimes x_{-1} \otimes x_0) &= a(h \otimes k_1 x_{-1} \lambda k_3 \otimes k_2 \cdot x_0) \\ \Leftrightarrow \chi\Big(\epsilon(x_{-1}) hk \otimes x_0\Big) &= \chi\Big(\epsilon(k_1 x_{-1} \lambda k_3) h \otimes k_2 \cdot x_0\Big) \\ \Leftrightarrow hk \cdot x &= h \cdot k \cdot x \ . \end{aligned}$$

Thus these conditions simply say that (X, χ) is a left H-module. Finally, the fact that a should be a morphism of left H-comodules rewrites as

$$pa = (\mathsf{id}_H \otimes a)(t \otimes^{H} \mathsf{id}_x) \Leftrightarrow pa(\mathsf{id}_H \otimes p) = (\mathsf{id}_H \otimes a)(t \otimes^{H} \mathsf{id}_x)(\mathsf{id}_H \otimes p)$$

$$\Leftrightarrow p\chi = (\mathsf{id}_H \otimes \chi)(\mathsf{id}_H \otimes \mathsf{id}_H \otimes \epsilon \otimes \mathsf{id}_X)(t \otimes \mathsf{id}_X)(\mathsf{id}_H \otimes p)$$

$$\Leftrightarrow p(h \cdot x) = (\mathsf{id}_H \otimes \chi)(\mathsf{id}_H \otimes \mathsf{id}_H \otimes \epsilon \otimes \mathsf{id}_X)(t(h \otimes x_{-1}) \otimes x_0) \ \forall \ h \otimes x \in H \otimes X$$

$$\Leftrightarrow (h \cdot x)_{-1} \otimes (h \cdot x)_0 = (\mathsf{id}_H \otimes \chi)(\mathsf{id}_H \otimes \mathsf{id}_H \otimes \epsilon \otimes \mathsf{id}_X)(h_1 x_{-2} \lambda(h_3) \otimes h_2 \otimes x_{-1} \otimes x_0)$$

$$\Leftrightarrow (h \cdot x)_{-1} \otimes (h \cdot x)_0 = (\mathsf{id}_H \otimes \chi)(h_1 x_{-1} \lambda(h_3) \otimes h_2 \otimes x_0)$$

$$\Leftrightarrow (h \cdot x)_{-1} \otimes (h \cdot x)_0 = h_1 x_{-1} \lambda(h_3) \otimes h_2 \cdot x_0 \ \forall\ h \in H,\ x \in X\ .$$

Now, this condition is equivalent to the Yetter-Drinfeld condition (YD): to imply (YD), tensor both sides with $k \in H$, then replace $h \otimes k$ by $\Delta(h) = h_1 \otimes h_2$, multiply the third and first coordinates and use the definition of antipode; to deduce it from (YD), tensor both sides of (YD) with $k \in H$, then replace $h \otimes k$ by $\Delta(h)$, apply λ to the third coordinate, multiply it to the first and use the definition of antipode.

Thus, a $\mathfrak{D}_H$-representation (X, p, a) is the same thing as a left Yetter-Drinfeld module (X, p, χ).

4. <u>$D(kG)$-modules.</u> Notice that the definition of $\mathfrak{D}_H$ makes sense for any Hopf monoid H in a symmetric monoidal category $\mathcal{S}$. In particular if G is a group then there is defined a category $\mathfrak{D}_G$ in Sets, which can be described in terms of pictures:

$$\begin{tikzcd}
g \arrow[r, bend left, "{(h,g)}"] \arrow[rr, bend right, "{(kh,g)}"'] & hgh^{-1} \arrow[r, bend left, "{(k,hgh^{-1})}"] & khgh^{-1}k^{-1} & & g \arrow[loop above, "{(1,g)}"] & .
\end{tikzcd}$$

This is the deltacategory arising from the double group corresponding to the action of G on itself by conjugation (section 8.1). In particular, $\mathsf{Rep}_{\mathit{Sets}}(\mathfrak{D}_G)$ is the category of crossed G-sets (a crossed G-set is a G-set X equipped with a map $|\ | : X \to G$ such that $|g \cdot x| = g|x|g^{-1}$). This is the set-theoretic analog of the description of $\mathsf{Rep}_k(\mathfrak{D}_H)$ as Yetter-Drinfeld modules.

Now, $\mathfrak{D}_G$ is an ordinary groupoid. Any small groupoid $\mathcal{G}$ is equivalent to the following disjoint union of groups (viewed as a groupoid)

$$\coprod_{x\in\pi_0(\mathcal{G})} \mathsf{Aut}_{\mathcal{G}}(x)\ ,$$

where $\pi_0(\mathcal{G})$ is the set of connected components of $\mathcal{G}$ and $\mathsf{Aut}_{\mathcal{G}}(x)$ is the automorphism group in $\mathcal{G}$ of (any) one object in the component x. For the groupoid in question this gives an equivalence

$$\mathfrak{D}_G \sim \coprod_{x\in\pi_0(G)} Z_G(x)$$

where $\pi_0(G)$ is the set of conjugacy classes of G and $Z_G(x)$ is the centralizer of any element in the class x. Applying the linearization functor $\mathit{Sets} \to \mathit{Vec}_k$ we obtain an equivalence

$$\mathfrak{D}_{kG} \sim \coprod_{x\in\pi_0(G)} kZ_G(x)$$

between linear categories. Passing to representations we obtain an equivalence

$$\{\text{ left Yetter-Drinfeld } kG\text{-modules }\} \sim \prod_{x\in\pi_0(G)} \{\ kZ_G(x)\text{-modules }\}\ .$$

A different proof of this result can be found in [CR1, proposition 3.7].

9.7 Internal categories and Hopf algebras

9.7.1 Smash products and biproducts

Let H be a k-bialgebra and A be a left H-comodule algebra. In other words, A is a monoid in the monoidal category $\mathsf{Comod}H$, or more explicitly, (A, μ_A, u_A) is a

k-algebra, (A, p_A) is a left H-comodule and

$$\mu_A : A{\otimes}A \to A \text{ and } \mathsf{u}_A : k \to A \text{ are morphisms of } H\text{-comodules,}$$

(where $A{\otimes}A$ and k are H-comodules by corestriction via μ_H and ϵ_H respectively), or equivalently,

$$p_A : A \to H{\otimes}A \text{ is a morphism of } k\text{-algebras .}$$

Write $p_A(a) = a_{-1}{\otimes}a_0$. Then there is a category $\underset{\vee}{\mathrm{A}} \rtimes \widehat{\mathrm{H}} = (A{\otimes}H, H, s, t, i, m)$, as follows:

$$\begin{array}{rclrcl} s : A{\otimes}H & \to & (A{\otimes}H){\otimes}H, & a{\otimes}h & \mapsto & a{\otimes}h_1{\otimes}h_2 \\ t : A{\otimes}H & \to & H{\otimes}(A{\otimes}H), & a{\otimes}h & \mapsto & a_{-1}h_1{\otimes}a_0{\otimes}h_2 \\ i : H & \to & A{\otimes}H, & h & \mapsto & 1{\otimes}h \\ m : (A{\otimes}H){\otimes}^H(A{\otimes}H) & \to & A{\otimes}H, & a{\otimes}h{\otimes}b{\otimes}k & \mapsto & \epsilon(h)ab{\otimes}k \end{array}$$

The category axioms (definition 2.3.1) are easily checked.

Since A is a monoid in $\mathsf{Comod}H$, the category $\mathsf{Mod}_{\mathsf{Comod}H}A$ is defined. An object of this category is a left H-comodule M which is also a left A-module in such a way that the action map $A{\otimes}M \to M$ is a morphism of left H-comodules (here $A{\otimes}M$ is viewed as left H-comodule by using the monoidal structure of $\mathsf{Comod}H$, that is, by corestriction via μ_H). These objects are sometimes called left Hopf (H, A)-modules [Mon, 8.5.1].

We claim that $\mathsf{Rep}_k(\underset{\vee}{\mathrm{A}} \rtimes \widehat{\mathrm{H}}) = \mathsf{Mod}_{\mathsf{Comod}H}A$. In fact, a $\underset{\vee}{\mathrm{A}} \rtimes \widehat{\mathrm{H}}$-representation is, by definition 6.1.1, a left H-comodule M, equipped with a morphism of left H-comodules $a : (A{\otimes}H){\otimes}^H M \to M$, which is associative and unital. Here, $A{\otimes}H$ is viewed as left H-comodule by means of t. Notice that this is precisely the structure obtained by corestriction via μ_H from its canonical left $H\otimes H$-comodule structure.

Now, by propositions 2.2.2 and 2.2.3 and the definition of s, $(A\otimes H)\otimes^{H}M \cong A\otimes M$. Moreover, it is easy to check that this is an isomorphism of left H-comodules, when both $A\otimes H$ and $A\otimes M$ are viewed as left H-comodules by corestriction via μ_H. Therefore, to give a morphism of left H-comodules $a : (A\otimes H)\otimes^{H}M \to M$ is equivalent to giving a morphism of left H-comodules $\tilde{a} : A\otimes M \to M$. Clearly, associativity and unitality for a correspond to those for $\tilde{a}$. Thus, a $\underset{\vee}{\mathrm{A}}\rtimes\widehat{\mathrm{H}}$-representation (M, p, a) is the same thing as an object $(M, p, \tilde{a})$ of $\mathsf{Mod}_{\mathsf{Comod}H}A$.

The algebra of admissible sections of $\underset{\vee}{\mathrm{A}}\rtimes\widehat{\mathrm{H}}$ is just a smash product in disguise. More precisely, it follows from remark 3.0.1 and the definition of s that, as k-spaces,

$$\Gamma(\underset{\vee}{\mathrm{A}}\rtimes\widehat{\mathrm{H}}) = \mathsf{Hom}_H(H, A\otimes H) \cong \mathsf{Hom}_k(H, A) \ .$$

If H is finite-dimensional, then

$$\Gamma(\underset{\vee}{\mathrm{A}}\rtimes\widehat{\mathrm{H}}) \cong \mathsf{Hom}_k(H, A) \cong A\otimes H^* \ .$$

Now, in this case, H^* is also a k-bialgebra, and A is a left H^{*op}-module algebra; hence, the smash product $A\#H^{*op}$ is defined [Mon, 4.1.3]. It is easy to see that the above is an isomorphism of k-algebras

$$\Gamma(\underset{\vee}{\mathrm{A}}\rtimes\widehat{\mathrm{H}}) \cong A\#H^{*op} \ .$$

For arbitrary H, the canonical inclusion

$$A\#H^{\circ op} \hookrightarrow \mathsf{Hom}_k(H, A) \cong \Gamma(\underset{\vee}{\mathrm{A}}\rtimes\widehat{\mathrm{H}})$$

is a morphism of k-algebras, where H° is the *finite* dual of H, as in [Mon, 1.2.3 and 9.1.1].

The question of when $\underset{\vee}{\mathrm{A}}\rtimes\widehat{\mathrm{H}}$ may be a deltacategory naturally arises. It is easy to see that this is the case if, in addition to being a left H-comodule algebra, A is also a left H-module coalgebra, and these structures are compatible in the sense that A is a left Yetter-Drinfeld H-module (section 9.6) and moreover a bimonoid in this category (bimonoids are defined in any *braided* monoidal category). In this case the deltacategory structure on $\underset{\vee}{\mathrm{A}}\rtimes\widehat{\mathrm{H}}$ is given by $\Delta_0 = \mu_H : H{\otimes}H \to H$ and

$$\Delta_1 : (A{\otimes}H){\otimes}^H(H{\otimes}H) \to (A{\otimes}H){\otimes}(A{\otimes}H), \quad (a{\otimes}hk){\otimes}(h{\otimes}k) \mapsto (a^1{\otimes}h_2){\otimes}(\lambda_H^{-1}h_1 \cdot a^2{\otimes}k)\ ,$$

and $\epsilon_0 = \mathsf{u}_H : k \to H$ and

$$\epsilon_1 : (A{\otimes}H){\otimes}^H k \to k, \quad a{\otimes}1 \mapsto \epsilon_A(a)\ ,$$

where $\Delta_A(a) = a^1{\otimes}a^2$ and ϵ_A are the comultiplication and counit of the coalgebra A. If H is finite-dimensional one then obtains a bialgebra structure on $\Gamma(\underset{\vee}{\mathrm{A}}\rtimes\widehat{\mathrm{H}}) \cong A\#H^{*op}$. This is non-other than the *biproduct* of [Mon, 10.6.5 or 10.6.15], or the *bosonization* of [Maj, 9.4.12]. For arbitrary H, $A\#H^{\circ op}$ becomes a bialgebra under the restriction of

$$\Gamma(\underset{\vee}{\mathrm{A}}\rtimes\widehat{\mathrm{H}}) \xrightarrow{\Gamma(\Delta)} \Gamma(\underset{\vee}{\mathrm{A}}\rtimes\widehat{\mathrm{H}}{\otimes}\underset{\vee}{\mathrm{A}}\rtimes\widehat{\mathrm{H}})\ .$$

9.7.2 Hopf modules

Let H be a k-bialgebra. There is a category $\mathfrak{M}_H = (H{\otimes}H, H, s, t, i, m)$ in Vec_k as follows:

$$\begin{array}{rcllcl} s : H{\otimes}H & \longrightarrow & (H{\otimes}H){\otimes}H, & a{\otimes}b & \mapsto & a{\otimes}b_1{\otimes}b_2 \\ t : H{\otimes}H & \longrightarrow & H{\otimes}(H{\otimes}H), & a{\otimes}b & \mapsto & a_1b_1{\otimes}a_2{\otimes}b_2 \\ i : H & \longrightarrow & H{\otimes}H, & a & \mapsto & 1{\otimes}a \\ m : (H{\otimes}H){\otimes}^H(H{\otimes}H) & \longrightarrow & H{\otimes}H, & a{\otimes}b{\otimes}c{\otimes}d & \mapsto & \epsilon(b)ac{\otimes}d \end{array}$$

In fact, $\mathfrak{M}_H = \underset{\vee}{\mathrm{H}} \rtimes \widehat{\mathrm{H}}$, the category described in section 9.7.1, where H is viewed as left H-comodule algebra via Δ_H. In particular, it follows that $\mathsf{Rep}_k\mathfrak{M}_H = \mathsf{Mod}_{\mathsf{Comod}H}H$. An object of this category is a left H-comodule M, which is also a left H-module in such a way that the action map $H{\otimes}M \to M$ is a morphism of left H-comodules, or equivalently, that the coaction map $H \to M{\otimes}H$ is a morphism of left H-modules (here $H{\otimes}M$ is viewed as left H-comodule and H-module by using the monoidal structures of $\mathsf{Comod}H$ and $\mathsf{Mod}H$ respectively). This is precisely the definition of a left Hopf H-module [Mon, 1.9.1]. Thus

$$\mathsf{Rep}_k\mathfrak{M}_H = \{\text{ left Hopf } H\text{-modules }\} \,.$$

We know from the general considerations for $\underset{\vee}{\mathrm{A}} \rtimes \widehat{\mathrm{H}}$ of section 9.7.1 that, when H is finite-dimensional, $\Gamma(\mathfrak{M}_H) \cong H\#(H^*)^{op}$, where H is viewed as left $(H^*)^{op}$-module algebra via $f \cdot a = f(a_1)a_2$ for $f \in H^*$ and $a \in H$. This algebra is sometimes called the *Heisenberg double* of H (for a slightly different version, see [Mon, 4.1.10]).

We will now present a new proof of the Fundamental Theorem on Hopf modules [Mon, 1.9.4]. The usual proof deals with the modules themselves. Instead, we will prove that there is an equivalence of internal categories $\mathfrak{M}_H \cong \underset{\vee}{\widehat{\mathrm{H}}} \sim \mathfrak{I}$. Passing to representations we then obtain the theorem.

We first consider representations of the pair category $\underset{\vee}{\widehat{\mathrm{H}}}$ (examples 2.4.1). Recall that the *one-arrow* category $\mathfrak{I} = (k, k, \dots)$ is such that $\mathsf{Rep}_k\mathfrak{I} = \mathit{Vec}_k$.

Since $\overrightarrow{\mathit{Cat}}_{\mathcal{S}}$ is a 2-category, the notion of equivalence of internal categories is defined. Explicitly, two internal categories $\mathfrak{C}$ and $\mathfrak{D}$ are equivalent if there are functors $f : \mathfrak{C} \to \mathfrak{D}$ and $g : \mathfrak{D} \to \mathfrak{C}$, and natural isomorphisms $\alpha : gf \Rightarrow \mathsf{id}_{\mathfrak{C}}$ and $\beta : fg \Rightarrow \mathsf{id}_{\mathfrak{D}}$. A natural isomorphism is a natural transformation that is invertible with respect to vertical composition. For the relevant definitions see section 4.1.

Lemma 9.7.1. *For any k-bialgebra H, $\underset{\vee}{\widehat{H}}$ and $\mathfrak{I}$ are equivalent as internal categories in Vec_k.*

Proof. Consider the functors $\overrightarrow{\epsilon} = (\epsilon_H \otimes \epsilon_H, \epsilon_H) : \underset{\vee}{\widehat{\mathrm{H}}} \to \mathfrak{I}$ and $\overrightarrow{u} = (\mathsf{u}_H \otimes \mathsf{u}_H, \mathsf{u}_H) : \mathfrak{I} \to \underset{\vee}{\widehat{\mathrm{H}}}$. We have $\overrightarrow{\epsilon}\,\overrightarrow{u} = \mathsf{id}_{\mathfrak{I}}$. On the other hand, one checks easily that the map $\mathsf{u}_H \otimes \mathsf{id}_H : H \to H \otimes H$ defines a natural isomorphism $\alpha : \overrightarrow{u}\,\overrightarrow{\epsilon} \Rightarrow \mathsf{id}_{\underset{\vee}{\widehat{\mathrm{H}}}}$, with inverse $\alpha^{-1} : \mathsf{id}_{\underset{\vee}{\widehat{\mathrm{H}}}} \Rightarrow \overrightarrow{u}\,\overrightarrow{\epsilon}$ defined by the map $\mathsf{id}_H \otimes \mathsf{u}_H : H \to H \otimes H$. ☐

As explained in section 6.2, passage to representations defines a 2-functor $\overrightarrow{\mathit{Cat}}_{\mathcal{S}} \to \mathit{LCat}$, $\mathfrak{C} \mapsto \mathsf{Rep}_{\mathcal{S}}\mathfrak{C}$. Since equivalences are preserved under 2-functors, it follows that, for any k-bialgebra H, there is an equivalence of large categories

$$\mathsf{res}_{\overrightarrow{\epsilon}} : \mathit{Vec}_k \to \mathsf{Rep}_k \underset{\vee}{\widehat{\mathrm{H}}} \ .$$

Side remark: notice that the proof of the lemma remains valid for any *coaugmented* k-coalgebra in place of H, that is a k-coalgebra C equipped with a morphism of coalgebras $u : k \to C$. Notice that if C possesses no coaugmentation, then there is no functor $\mathfrak{I} \to \widehat{\underset{\vee}{\mathrm{C}}}$. However, one can show that $\mathsf{res}_{\overrightarrow{\epsilon}} : \mathit{Vec}_k \sim \mathsf{Rep}_k\widehat{\underset{\vee}{\mathrm{C}}}$ is still an equivalence for *any* k-coalgebra C. We omit the proof of this fact. If C is a finite-dimensional k-coalgebra, then we have, from section 5.2 and proposition 9.2.4 ,

$$\mathsf{Mod}(\mathsf{End}_k C) = \mathsf{Mod}\Gamma(\widehat{\underset{\vee}{\mathrm{C}}}) \cong \mathsf{Rep}_k\widehat{\underset{\vee}{\mathrm{C}}} \sim \mathit{Vec}_k \ ;$$

this simply says that the matrix ring $\mathsf{End}_k(C)$ is Morita-equivalent to k.

Proposition 9.7.1. *For any Hopf k-algebra H, $\widehat{\underset{\vee}{\mathrm{H}}}$ and $\mathfrak{M}_H$ are isomorphic as internal categories in Vec_k.*

Proof. Consider the functors $f : \mathfrak{M}_H \to \widehat{\underset{\vee}{\mathrm{H}}}$ defined by $f_0 = \mathsf{id}_H : H \to H$ and $f_1 : H{\otimes}H \to H{\otimes}H$, $f_1(a{\otimes}b) = ab_1{\otimes}b_2$, and $g : \widehat{\underset{\vee}{\mathrm{H}}} \to \mathfrak{M}_H$ defined by $g_0 = \mathsf{id}_H : H \to H$ and $g_1 : H \to H$, $g_1(a{\otimes}b) = a\lambda b_1{\otimes}b_2$, where λ is the antipode of H. Then f and g are inverse functors. The verification of these assertions only amounts to routine use of the Hopf algebra axioms for H. $\square$

As a corollary we obtain the Fundamental Theorem on Hopf modules:

Corollary 9.7.1. *For any Hopf k-algebra H, the functor*

$$\mathit{Vec}_k \to \{\textit{left Hopf } H\textit{-modules}\}, \quad V \mapsto H{\otimes}V \ ,$$

is an equivalence. Here $H{\otimes}V$ is viewed as left H-comodule via $\Delta_H{\otimes}\mathit{id}_V$ and as left H-module via $\mu_H{\otimes}\mathit{id}_V$.

Proof. Composing the isomorphism $f : \mathfrak{M}_H \xrightarrow{\cong} \underset{\vee}{\underline{\widehat{\mathrm{H}}}}$ with the equivalence $\overrightarrow{\epsilon} : \underset{\vee}{\underline{\widehat{\mathrm{H}}}} \xrightarrow{\sim} \mathfrak{I}$ we obtain an equivalence $\overrightarrow{\epsilon} f : \mathfrak{M}_H \xrightarrow{\sim} \mathfrak{I}$. Passing to representations we obtain an equivalence

$$\mathsf{res}_{\overrightarrow{\epsilon} f} : \mathit{Vec}_k \xrightarrow{\sim} \mathsf{Rep}_{\mathit{Vec}_k}(\mathfrak{M}_H) \ ,$$

which by definition of restriction along functors (section 6.2) has the announced form $V \to H{\otimes}V$. ☐

We also obtain for free the result of corollary 9.4.3 in [Mon]:

Corollary 9.7.2. *For any finite-dimensional Hopf k-algebra H,*

$$H\#(H^*)^{op} \cong \mathit{End}_k H$$

as k-algebras.

Proof. The isomorphism $f : \mathfrak{M}_H \to \underset{\vee}{\underline{\widehat{\mathrm{H}}}}$ can be seen as a cofunctor by remark 4.2.1. Hence, by proposition 5.3.1, there is a corresponding isomorphism of k-algebras

$$H\#(H^*)^{op} = \Gamma(\mathfrak{M}_H) \xrightarrow{\cong} \Gamma(\underset{\vee}{\underline{\widehat{\mathrm{H}}}}) = \mathsf{End}_k(H) \ .$$

☐

9.7.3 Categories associated to a Hopf algebra

Let H be a Hopf k-algebra. As we have seen, there are several categories in Vec_k naturally associated to H. We summarize them in table 9.1.

Table 9.1: Categories associated to a Hopf algebra H

Category	Representations	Admissible sections	(if H is finite dimensional)
$\widehat{\mathrm{H}}$	left H-comodules	$(H^*)^{op}$	
$\underset{\vee}{\mathrm{H}}$	left H-modules	H	
$\underset{\vee}{\widehat{\mathrm{H}}}$	equivalent to $\boldsymbol{Vec}_k$ under $V \mapsto H \otimes V$	$\mathsf{End}_k(H)$ under composition	
$\underset{\vee}{\mathrm{H}} \otimes \widehat{\mathrm{H}}$		$\mathsf{End}_k(H)$ under convolution	$H \otimes (H^*)^{op}$ (usual tensor product)
$\mathfrak{M}_H$	left Hopf H-modules		$H \# (H^*)^{op}$ (Heisenberg double)
$\mathfrak{D}_H$	left Yetter-Drinfeld H-modules		$D(H)$ (Drinfeld's double)
$\mathfrak{B}_H$	Hopf H-bimodules		see below

The category $\underset{\vee}{\mathrm{H}}\otimes\widehat{\mathrm{H}}$ is a special case of that one discussed in example 7.2.1. It is also the category $\underset{\vee}{\mathrm{A}}\rtimes\widehat{\mathrm{H}}$ of section 9.7.1, for the special case when $A = H$ viewed as trivial H-comodule k-algebra (i.e. via $\mathsf{u}_H\otimes\mathsf{id}_A : H \to H\otimes A$).

The only category that remains to be discussed is $\mathfrak{B}_H$. Its definition is as follows: $\mathfrak{B}_H = (H\otimes H\otimes H\otimes H, H\otimes H, s, t, i, m)$, where

$$\begin{aligned} s : H^{\otimes 4} \longrightarrow (H^{\otimes 4})\otimes(H^{\otimes 2}), &\quad g\otimes h\otimes a\otimes b \mapsto g\otimes h\otimes a_1\otimes b_1\otimes a_2\otimes b_2 \\ t : H^{\otimes 4} \longrightarrow (H^{\otimes 2})\otimes(H^{\otimes 4}), &\quad g\otimes h\otimes a\otimes b \mapsto g_1 a_1 h_1\otimes g_2 b_1 h_2\otimes g_3\otimes h_3\otimes a_2\otimes b_2 \\ i : H^{\otimes 2} \longrightarrow H^{\otimes 4}, &\quad a\otimes b \mapsto 1\otimes 1\otimes a\otimes b \\ m : (H^{\otimes 4})\otimes_{H^{\otimes 2}}(H^{\otimes 4}) \longrightarrow H^{\otimes 4}, &\quad (f\otimes k\otimes c\otimes d)\otimes(g\otimes h\otimes a\otimes b) \mapsto \epsilon(cd)fg\otimes hk\otimes a\otimes b\ . \end{aligned}$$

The set-theoretic analog of $\mathfrak{B}_H$ in table 9.2 may be helpful in order to grasp this definition.

One checks easily that $\mathsf{Rep}_k\mathfrak{B}_H$ is the category of Hopf H-bimodules, as in [CR1] or [Ros]. Rosso proves in [Ros] that this category is equivalent to that of Yetter-Drinfeld modules, by arguing directly with the modules themselves. As in the case of the Fundamental Theorem on Hopf modules, it is possible to make use of the notion of internal categories to obtain an alternative proof. Namely, there is an isomorphism of internal categories $\mathfrak{B}_H \cong \mathfrak{D}_H\otimes\mathfrak{M}_H$ defined by the functors $f : \mathfrak{B}_H \to \mathfrak{D}_H\otimes\mathfrak{M}_H$ and $g : \mathfrak{D}_H\otimes\mathfrak{M}_H \to \mathfrak{B}_H$ given by

$$\begin{array}{rclrcl} f_0 : H\otimes H & \to & H\otimes H, & a\otimes b & \mapsto & a\lambda(b_1)\otimes b_2 \\ f_1 : H\otimes H\otimes H\otimes H & \to & (H\otimes H)\otimes(H\otimes H), & g\otimes h\otimes a\otimes b & \mapsto & g_1\otimes a\lambda(b_1)\otimes g_2 b_2 h\lambda(b_3)\otimes b_4 \\ g_0 : H\otimes H & \to & H\otimes H, & a\otimes b & \mapsto & ab_1\otimes b_2 \\ g_1 : (H\otimes H)\otimes(H\otimes H) & \to & H\otimes H\otimes H\otimes H, & g\otimes a\otimes h\otimes b & \mapsto & g_1\otimes\lambda(b_1)\lambda(g_2)hb_2\otimes ab_3\otimes b_4 \end{array}$$

The verification of the relevant axioms is lengthy but straightforward.

Together with the equivalence $\mathfrak{M}_H \sim \mathfrak{J}$ of section 9.7.2, we obtain an equivalence $\mathfrak{B}_H \sim \mathfrak{D}_H$. Passing to representations yields the announced result.

Cibils and Rosso have constructed, for any finite-dimensional Hopf algebra H, an algebra X such that $\mathsf{Mod}X$ is the category of Hopf H-bimodules [CR2]. From our point of view this requires no additional work: $X = \Gamma(\mathfrak{B}_H)$ has this property, by proposition 9.2.4 (it applies since $H^{\otimes 4}$ is free as right $H^{\otimes 2}$-comodule, hence flat). Moreover, since $\mathfrak{B}_H \cong \mathfrak{D}_H {\scriptstyle\otimes} \mathfrak{M}_H$, we also have that $X \cong D(H){\scriptstyle\otimes}(H\#(H^*)^{op})$. This result is obtained by other means in [CR2].

The definition of all the categories in table 9.1 makes sense in any symmetric monoidal category $\mathcal{S}$ in place of $\mathtt{Vec}_k$. In particular they make sense in $\mathcal{S} = \mathtt{Sets}$, replacing the Hopf k-algebra H for a group G. In this case those categories can be described by means of pictures, as in table 9.2.

These pictures serve as good guides when finding relationships among the various categories. For instance, the functors $f : \mathfrak{M}_H \to \underset{\vee}{\widehat{\mathrm{H}}}$ and $g : \underset{\vee}{\widehat{\mathrm{H}}} \to \mathfrak{M}_H$ of proposition 9.7.1 can be described in the set-theoretic case through the pictures

$$\begin{tikzcd}
\mathfrak{M}_H \arrow[d, "f"'] & b \arrow[rr, bend left, "{(a,b)}"] & {} \arrow[d, mapsto] & ab & & \mathfrak{M}_H & b \arrow[rr, bend left, "{(ab^{-1},b)}"] & {} & a\ . \\
\underset{\vee}{\widehat{\mathrm{H}}} & b \arrow[rr, bend left, "{(ab,b)}"] & {} & ab & & \underset{\vee}{\widehat{\mathrm{H}}} \arrow[u, "g"] & b \arrow[rr, bend left, "{(a,b)}"] & {} \arrow[u, mapsto] & a
\end{tikzcd}$$

These set-theoretic analogies are seldom available when one ignores internal categories and restricts attention to algebras of admissible sections, because of the lack of duality in $\mathtt{Sets}$. For instance, the set-theoretic analog of $\mathfrak{M}_H$ is $\mathfrak{M}_G$, but

Table 9.2: Categories associated to a group G

Category	Composition	Identities
$\widehat{\mathrm{G}}$	(only identities)	$\begin{tikzcd} a \arrow[loop above, "a"] \end{tikzcd}$
$\underset{\vee}{\mathrm{G}}$	$\begin{tikzcd} * \arrow[r, bend left, "b"] \arrow[rr, bend right, "ab"'] & * \arrow[r, bend left, "a"] & * \end{tikzcd}$	$\begin{tikzcd} * \arrow[loop above, "1"] \end{tikzcd}$
$\underset{\vee}{\widehat{\mathrm{G}}}$	$\begin{tikzcd} c \arrow[r, bend left, "(b{,}c)"] \arrow[rr, bend right, "(a{,}c)"'] & b \arrow[r, bend left, "(a{,}b)"] & a \end{tikzcd}$	$\begin{tikzcd} a \arrow[loop above, "(a{,}a)"] \end{tikzcd}$
$\underset{\vee}{\mathrm{G}} \times \widehat{\mathrm{G}}$	$\begin{tikzcd} b \arrow[r, bend left, "(c{,}b)"] \arrow[rr, bend right, "(ac{,}b)"'] & b \arrow[r, bend left, "(a{,}b)"] & b \end{tikzcd}$	$\begin{tikzcd} a \arrow[loop above, "(1{,}a)"] \end{tikzcd}$
$\mathfrak{M}_G$	$\begin{tikzcd} d \arrow[r, bend left, "(c{,}d)"] \arrow[rr, bend right, "(ac{,}d)"'] & cd \arrow[r, bend left, "(a{,}cd)"] & acd \end{tikzcd}$	$\begin{tikzcd} a \arrow[loop above, "(1{,}a)"] \end{tikzcd}$
$\mathfrak{D}_G$	$\begin{tikzcd} d \arrow[r, bend left, "(c{,}d)"] \arrow[rr, bend right, "(ac{,}d)"'] & cdc^{-1} \arrow[r, bend left, "(a{,}cdc^{-1})"] & acdc^{-1}a^{-1} \end{tikzcd}$	$\begin{tikzcd} a \arrow[loop above, "(1{,}a)"] \end{tikzcd}$
$\mathfrak{B}_G$	$\begin{tikzcd} (a,b) \arrow[r, bend left, "(g{,}h{,}a{,}b)"] \arrow[rr, bend right, "(fg{,}hk{,}a{,}b)"'] & (gah,gbh) \arrow[r, bend left, "(f{,}k{,}gah{,}gbh)"] & (fgahk,fgbhk) \end{tikzcd}$	$\begin{tikzcd} (a,b) \arrow[loop above, "(1{,}1{,}a{,}b)"] \end{tikzcd}$

the Heisenberg double $H\#(H^*)^{op}$ has no good set-theoretic analog.

9.8 Binomial braids, $U_q^+(g)$ as admissible sections

In this section we describe a general procedure for constructing a quantum group $U_H^+(X)$ out of a Hopf algebra H and a Yetter-Drinfeld H-module, as admissible sections of a certain deltacategory $\mathfrak{U}_H^+(X)$ in Vec_k. The quantum groups of Drinfeld, Jimbo and Lusztig are obtained through this procedure from the simplest choice of H: group algebras $H = kG$ of cyclic groups G. In this procedure, the action of the *binomial braids* (appendix B) $b_i^{(n)}$ on the various tensor powers $X^{\otimes n}$ plays a crucial role.

9.8.1 Preliminaries on braids

The braid groups B_n and the braid category $\mathfrak{B}$ are defined in appendix B. The definitions of the first three sections of this appendix are needed for all that follows. Results from later sections will be quoted as used. In particular, in section B.2.5 it is explained how Yang-Baxter operators yield monoidal representations of the braid category $\mathfrak{B}$. In appendix B only one-dimensional representations are considered, corresponding to the Yang-Baxter operator that simply multiplies by q.

An equivalent way to describe monoidal representations of the braid category is by means of the following fact: $\mathfrak{B}$ is the free braided monoidal strict category on one object (the object $1 \in \mathbb{N}$) [K, XIII.3.8]. This says that given any object X of a braided monoidal category $\mathcal{K}$, there is a unique functor $F : \mathfrak{B} \to \mathcal{K}$ that preserves the monoidal structures and the braidings and such that $F(1) = X$. If $\mathcal{K}$ carries in addition a k-linear structure (compatible with the rest of the structure), then F extends to the linearization of $\mathfrak{B}$, $F : k\mathfrak{B} \to \mathcal{K}$. Usually $\mathcal{K}$ consists of vector k-spaces with some additional structure, and thus $F : k\mathfrak{B} \to \mathcal{K}$ yields linear representations of the various braid groups.

A family of examples arises from the categories $\mathcal{K} = \mathsf{Rep}_k\mathfrak{D}_{kG}$ of Yetter-Drinfeld kG-modules (section 9.6) for any group G. An object of this category is a k-space X equipped with a linear action of G and a decomposition $X = \oplus_{g\in G} X_g$ into subspaces, such that the action of $h \in G$ carries X_g to $X_{hgh^{-1}}$. In this context, one usually writes $|x| = g$ when $x \in X_g$, so that the condition just mentioned becomes $|h \cdot x| = h|x|h^{-1}$. Since $\mathfrak{D}_{kG}$ is a deltacategory, $\mathsf{Rep}_k\mathfrak{D}_{kG}$ is monoidal. Explicitly, the tensor product $X \otimes Y$ of two representations is equipped with the G-action $g \cdot (x, y) = (g \cdot x, g \cdot y)$ and the G-grading $|(x, y)| = |x||y|$. Moreover, this category is braided, with braiding

$$\beta_{Y,X} : Y \otimes X \to X \otimes Y, \quad \beta_{Y,X}(y \otimes x) = x \otimes |x|^{-1} \cdot y \ .$$

(This braiding will be more convenient for us than the more usual $x \otimes y \mapsto (|x| \cdot y) \otimes x$). In particular, for each $n \geq 0$, $X^{\otimes n}$ is again a Yetter-Drinfeld KG-module, with

$$|x_1 \otimes \ldots \otimes x_n| = |x_1| \ldots |x_n| \text{ and } g \cdot (x_1 \otimes \ldots x_n) = (g \cdot x_1) \otimes \ldots (g \cdot x_n) \ ;$$

the braid group B_n acts on $X^{\otimes n}$ by morphisms of left Yetter-Drinfeld kG-modules, and in such a way that $\forall\ x \in X^{\otimes n}$, $y \in X^{\otimes m}$, $s \in B_n$ and $t \in B_m$,

$$(s \otimes t)(x \otimes y) = (sx) \otimes (ty) \ \text{ and } \ \beta_{m,n} : X^{\otimes m} \otimes X^{\otimes n} \to X^{\otimes n} \otimes X^{\otimes m} \text{ is } \beta_{m,n}(y \otimes x) = x \otimes |x|^{-1} \cdot y \ ,$$

where $\beta_{m,n} \in B_{m+n}$ is the *braiding* of appendix B.

Some results about the actions of the binomial braids $b_i^{(n)}$ on the tensor powers $X^{\otimes n}$ will be obtained in section 9.8.5. Recall from the appendix that if X is one-dimensional, and $s_1^{(2)}$ acts on $X \otimes X$ by multiplication by q, then $b_i^{(n)}$ acts on $X^{\otimes n}$ by multiplication by the q-binomial coefficient $\left[{n \atop i} \right]_q$.

9.8.2 Binomial deltacategories

Let G be a group and X a left Yetter-Drinfeld kG-module, with G-grading $|\ | : X \to G$ and G-action $(g, x) \mapsto g \cdot x$.

Consider the linear graph $\mathfrak{G}_G(X)$ with objects G and set of arrows $\left(\coprod_{g \in G} X_g\right) \times G$, where (x, g) is an arrow from g to $|x|g$:

$$\begin{tikzcd} g \arrow[r, bend left, "{(x,g)}"] & {|x|g} \end{tikzcd}$$

As a graph in $\textbf{\textit{Vec}}_k$, $\mathfrak{G}_G(X) = (X \otimes kG, kG)$. Let $\mathfrak{T}_G(X)$ be the free category on this graph, as in section 9.4. Thus $\mathfrak{T}_G(X) = (A, kG, s, t, i, m)$ where

$$\begin{aligned} A = \perp^{kG}(X \otimes kG) &= kG \oplus (X \otimes kG) \oplus \Big((X \otimes kG) \otimes^{kG} (X \otimes kG) \Big) \oplus \dots \\ &\cong kG \oplus (X \otimes kG) \oplus (X \otimes X \otimes kG) \oplus \dots = T(X) \otimes kG, \end{aligned}$$

where $T(X)$ is the usual tensor k-algebra of the space X (we have used proposition

2.2.3), and

$$\begin{array}{rcllcl} s : A & \rightarrow & A\otimes kG, & x\otimes g & \mapsto & x\otimes g\otimes g \\ t : A & \rightarrow & kG\otimes A, & x\otimes g & \mapsto & |x|g\otimes x\otimes g \\ i : kG & \rightarrow & A, & g & \mapsto & 1\otimes g \\ m : A\otimes^{kG}A & \rightarrow & A, & (y\otimes|x|g)\otimes(x\otimes g) & \mapsto & y\otimes x\otimes g; \end{array}$$

or, in pictures,

$$\begin{tikzcd} g \arrow[r, bend left, "x\otimes g"] \arrow[rr, bend right, "yx\otimes g"'] & {|x|g} \arrow[r, bend left, "y\otimes|x|g"] & {|y||x|g} & g \arrow[loop above, "1\otimes g"] \end{tikzcd} .$$

Notice that the G-grading on the tensor powers of X described above endows $T(X)$ with a structure of left kG-comodule algebra, and that $\mathfrak{T}_G(X)$ is non-other than the category $\underset{\vee}{\mathrm{T}}\rtimes\widehat{\mathrm{H}}$ of section 9.7.1 (for $T = T(X)$, $H = kG$).

We claim that $\mathfrak{T}_G(X)$ is a deltacategory. There are two ways to proceed at this point. We could show that $T(X)$ is indeed a bimonoid in the category of left Yetter-Drinfeld kG-modules, and appeal to the remark about deltacategory structures on categories of the form $\underset{\vee}{\mathrm{T}}\rtimes\widehat{\mathrm{H}}$ in section 9.7.1. Instead, we choose to proceed directly, as follows.

We first define cofunctors of graphs $\Delta : \mathfrak{G}_G(X) \rightarrow \mathfrak{T}_G(X)\otimes\mathfrak{T}_G(X)$ and $\epsilon : \mathfrak{G}_G(X) \rightarrow \mathfrak{I}$, by

$$\begin{array}{rclcrcl} \Delta_0 : kG\otimes kG & \rightarrow & kG & \text{and} & \Delta_1 : (X\otimes kG)\otimes^{kG}(kG\otimes kG) & \rightarrow & A\otimes A \\ g\otimes h & \mapsto & gh & & (x\otimes gh)\otimes(g\otimes h) & \mapsto & (x\otimes g)\otimes(1\otimes h) \\ & & & & & & +(1\otimes g)\otimes(g^{-1}\cdot x\otimes h) \end{array}$$

and

$$\begin{array}{rcl} \epsilon_0 : k & \longrightarrow & kG \\ 1 & \longmapsto & 1 \end{array} \quad \text{and} \quad \begin{array}{rcl} \epsilon_1 : (X \otimes kG) \otimes^{kG} k & \longrightarrow & k \\ x \otimes 1 & \longmapsto & 0 \end{array} .$$

Here, and below, $g^{-1} \cdot x \otimes h$ means $(g^{-1} \cdot x) \otimes h$; also, we sometimes identify $k(G \times G) \cong kG \otimes kG$ via $(g, h) \mapsto g \otimes h$. Notice that the target of $(1 \otimes g) \otimes (g^{-1} \cdot x \otimes h)$ is $(|1|g, |g^{-1} \cdot x|h) = (g, g^{-1}|x|gh)$ (by definition of Yetter-Drinfeld module), which maps by Δ_0 to $|x|gh$, the target of $x \otimes gh$, so Δ preserves targets as required in the definition of cofunctor.

By proposition 9.4.2, Δ and ϵ extend to cofunctors (of categories) $\Delta : \mathfrak{T}_G(X) \to \mathfrak{T}_G(X) \otimes \mathfrak{T}_G(X)$ and $\epsilon : \mathfrak{T}_G(X) \to \mathfrak{I}$. By the uniqueness in proposition 9.4.2, it is enough to check coassociativity and counitality for Δ and ϵ on the generating graph $\mathfrak{G}_G(X)$. Arguing along the same lines as in section 9.5, to obtain coassociativity we need to show the equality between the two lifts of an arrow $x \otimes ghk$ to (g, h, k), via $(\Delta \otimes \mathsf{id}) \circ \Delta$ and $(\mathsf{id} \otimes \Delta) \circ \Delta$. In order to do this, notice that we have

$$\Delta_1\Big((1 \otimes gh) \otimes (g \otimes h)\Big) = (1 \otimes g) \otimes (1 \otimes h) \quad \text{and} \quad \epsilon_1(1 \otimes 1) = 1 \ ,$$

since by construction Δ and ϵ preserve identities. Now the lifts in question are:

$$\begin{tikzcd}
\mathfrak{T}\otimes\mathfrak{T}\otimes\mathfrak{T} & (g,h,k) \arrow[d, mapsto] & \begin{array}{c}\Big((x\otimes g)\otimes(1\otimes h)+(1\otimes g)\otimes(g^{-1}\cdot x\otimes h)\Big)\otimes(1\otimes k)+\\ +(1\otimes g)\otimes(1\otimes h)\otimes(h^{-1}g^{-1}\cdot x\otimes k)\end{array} \\
\mathfrak{T}\otimes\mathfrak{T} \arrow[u, "\Delta\otimes\mathrm{id}"] & (gh,k) \arrow[d, mapsto] & (x\otimes gh)\otimes(1\otimes k)+(1\otimes gh)\otimes(h^{-1}g^{-1}\cdot x\otimes k) \arrow[u, mapsto] \\
\mathfrak{T} \arrow[u, "\Delta"] & ghk & x\otimes ghk \arrow[u, mapsto]
\end{tikzcd} .$$

$$\begin{tikzcd}
\mathfrak{T}\otimes\mathfrak{T}\otimes\mathfrak{T} & (g,h,k) \arrow[d, mapsto] & \begin{array}{c}(x\otimes g)\otimes(1\otimes h)\otimes(1\otimes k)+\\ +(1\otimes g)\otimes\Big((g^{-1}\cdot x\otimes h)\otimes(1\otimes k)+(1\otimes h)\otimes(h^{-1}g^{-1}\cdot x\otimes k)\Big)\end{array} \\
\mathfrak{T}\otimes\mathfrak{T} \arrow[u, "\mathrm{id}\otimes\Delta"] & (g,hk) \arrow[d, mapsto] & (x\otimes g)\otimes(1\otimes hk)+(1\otimes g)\otimes(g^{-1}\cdot x\otimes hk) \arrow[u, mapsto] \\
\mathfrak{T} \arrow[u, "\Delta"] & ghk & x\otimes ghk \arrow[u, mapsto]
\end{tikzcd}$$

Thus, the two lifts agree as required. Counitality can be checked similarly. This completes the proof of the claim that $\mathfrak{T}_G(X)$ is a deltacategory.

Before introducing some relations in $\mathfrak{T}_G(X)$, we need to describe the cofunctor Δ explicitly. Recall from section 9.8.1 that, for each $n \geq 0$, the braid group B_n acts on $X^{\otimes n}$ in such a way that, $\forall\ x \in X^{\otimes n}$, $y \in X^{\otimes m}$, $s \in B_n$ and $t \in B_m$,

$$(s\otimes t)(x\otimes y) = (sx)\otimes(ty) \quad \text{and} \quad \beta_{m,n} : X^{\otimes m}\otimes X^{\otimes n} \to X^{\otimes n}\otimes X^{\otimes m} \text{ is } \beta_{m,n}(y\otimes x) = x\otimes|x|^{-1}\cdot y\ ,$$

where $\beta_{m,n} \in B_{m+n}$ is the *braiding* of appendix B. Moreover, the action of B_n on $X^{\otimes n}$ is by morphisms of left Yetter-Drinfeld modules, that is, it commutes with the G-grading and the G-action on $X^{\otimes n}$.

We claim that, for any $x \in X^{\otimes n}$ and $g, h \in G$,

$$\Delta_1\Big((x{\otimes}gh){\otimes}(g{\otimes}h)\Big) =$$

$$= \sum_{i=0}^{n}\Big((b_i^{(n)}x)^{(i)}{\otimes}g\Big){\otimes}\Big(g^{-1}\cdot(b_i^{(n)}x)^{(i)'}{\otimes}h\Big) \in \bigoplus_{i=0}^{n}(X^{\otimes i}{\otimes}kG){\otimes}(X^{\otimes(n-i)}{\otimes}kG) \subseteq A{\otimes}A\ .$$

Here $b_i^{(n)}$ is the binomial braid of appendix B; we have also used the notation $y = y^{(i)}{\otimes}y^{(i)'}$ for the canonical identification $X^{\otimes n} \cong X^{\otimes i}{\otimes}X^{\otimes(n-i)}$.

We will prove this claim by induction on n. For $n = 0$ it boils down to $\Delta_1\Big(1{\otimes}gh{\otimes}(g{\otimes}h)\Big) = (1{\otimes}g){\otimes}(1{\otimes}h)$, which holds as already mentioned. For $n = 1$ it reduces to the definition of Δ_1 on the generating arrows:

$$\Delta_1\Big((x{\otimes}gh){\otimes}(g{\otimes}h)\Big) = (1{\otimes}g){\otimes}(g^{-1}\cdot x{\otimes}h) + (x{\otimes}g){\otimes}(1{\otimes}h)\ .$$

Thus, it is enough to prove that if the claim holds for $x \in X$ and $y \in X^{\otimes n}$ with $n \geq 1$, then it does for $y{\otimes}x \in X^{\otimes(n+1)}$ too. Now, by definition of composition in $\mathfrak{T}_G(X)$, $y{\otimes}x{\otimes}gh$ is the composite of $x{\otimes}gh$ and $y{\otimes}|x|gh$. Since by construction Δ preserves compositions, we can compute $\Delta_1\Big((y{\otimes}x{\otimes}gh){\otimes}(g{\otimes}h)\Big)$ by first lifting $x{\otimes}gh$ to (g, h), then lifting $y{\otimes}|x|gh$ to the targets of these arrows, and composing.

The lift of $x{\otimes}gh$ is $(1{\otimes}g){\otimes}(g^{-1}\cdot x{\otimes}h) + (x{\otimes}g){\otimes}(1{\otimes}h)$. The targets of these two arrows of $\mathfrak{T}_G(X){\otimes}\mathfrak{T}_G(X)$ are $(g, g^{-1}|x|gh)$ and $(|x|g, h)$. The lifts of $y{\otimes}|x|gh$ to these targets are, by induction hypothesis,

$$\sum_{j=0}^{n}\Big((b_j^{(n)}y)^{(j)}{\otimes}g\Big){\otimes}\Big(g^{-1}\cdot(b_j^{(n)}y)^{(j)'}{\otimes}g^{-1}|x|gh\Big)$$

and

$$\sum_{j=0}^{n}\Big((b_j^{(n)}y)^{(j)}\otimes|x|g\Big)\otimes\Big(g^{-1}|x|^{-1}\cdot(b_j^{(n)}y)^{(j)'}\otimes h\Big)\ .$$

Composing appropiately we find the lift of $y\otimes x\otimes gh$, it is

$$\sum_{j=0}^{n}\Big((b_j^{(n)}y)^{(j)}\otimes g\Big)\otimes\Big(g^{-1}\cdot(b_j^{(n)}y)^{(j)'}\otimes g^{-1}\cdot x\otimes h\Big)+$$

$$+\sum_{j=0}^{n}\Big((b_j^{(n)}y)^{(j)}\otimes x\otimes g\Big)\otimes\Big(g^{-1}|x|^{-1}\cdot(b_j^{(n)}y)^{(j)'}\otimes h\Big)$$

$$=(1\otimes g)\otimes(g^{-1}\cdot y\otimes g^{-1}\cdot x\otimes h)+\sum_{j=1}^{n}\Big((b_j^{(n)}y)^{(j)}\otimes g\Big)\otimes\Big(g^{-1}\cdot(b_j^{(n)}y)^{(j)'}\otimes g^{-1}\cdot x\otimes h\Big)+$$

$$+\sum_{j=0}^{n-1}\Big((b_j^{(n)}y)^{(j)}\otimes x\otimes g\Big)\otimes\Big(g^{-1}|x|^{-1}\cdot(b_j^{(n)}y)^{(j)'}\otimes h\Big)+(y\otimes x\otimes g)\otimes(1\otimes h)\ ,$$

(since $b_0^{(n)}=b_n^{(n)}=\mathsf{id}$). On the other hand, we want to show that this lift is equal to

$$\sum_{i=0}^{n+1}\Big((b_i^{(n+1)}(y\otimes x))^{(i)}\otimes g\Big)\otimes\Big(g^{-1}\cdot(b_i^{(n+1)}(y\otimes x))^{(i)'}\otimes h\Big)=(1\otimes g)\otimes\Big(g^{-1}\cdot(y\otimes x)\Big)\otimes h+$$

$$+\sum_{i=1}^{n}\Big((b_i^{(n+1)}(y\otimes x))^{(i)}\otimes g\Big)\otimes\Big(g^{-1}\cdot(b_i^{(n+1)}(y\otimes x))^{(i)'}\otimes h\Big)+(y\otimes x\otimes g)\otimes(1\otimes h)\ .$$

Comparing these two expressions we see that it is enough to prove the following equality between elements of $X^{\otimes(n+1)}$:

$$\sum_{i=1}^{n}(b_i^{(n+1)}(y{\otimes}x))^{(i)}{\otimes}(b_i^{(n+1)}(y{\otimes}x))^{(i)'} \overset{(?)}{=} \sum_{j=1}^{n}(b_j^{(n)}y)^{(j)}{\otimes}(b_j^{(n)}y)^{(j)'}{\otimes}x +$$

$$+\sum_{j=0}^{n-1}(b_j^{(n)}y)^{(j)}{\otimes}x{\otimes}|x|^{-1}\cdot(b_j^{(n)}y)^{(j)'} \ .$$

Recalling the expression for the action of the braid β, we see that this is in turn equivalent to

$$\sum_{i=1}^{n} b_i^{(n+1)}(y{\otimes}x) \overset{(?)}{=} \sum_{j=1}^{n}(b_j^{(n)}y){\otimes}x + \sum_{j=0}^{n-1}(b_j^{(n)}y)^{(j)}{\otimes}\beta_{n-j,1}\Big((b_j^{(n)}y)^{(j)'}{\otimes}x\Big)$$

$$= \sum_{j=1}^{n}(b_j^{(n)}{\otimes}1)(y{\otimes}x) + \sum_{j=0}^{n-1}(1^{(j)}{\otimes}\beta_{n-j,1})(b_j^{(n)}{\otimes}1)(y{\otimes}x) \ .$$

Since

$$1^{(j)}{\otimes}\beta_{n-j,1} = 1^{(j)}{\otimes}s^{(n-j+1)}(1, n-j+1) = s^{(n+1)}(j+1, n+1)$$

(by equations 12 and 3 in appendix B), the equality in question is implied by

$$\sum_{i=1}^{n} b_i^{(n+1)} = \sum_{j=1}^{n} b_j^{(n)}{\otimes}1 + \sum_{j=0}^{n-1} s^{(n+1)}(j+1, n+1)(b_j^{(n)}{\otimes}1) \ ,$$

which indeed holds by Pascal's identity (equation 14 in appendix B). This completes the proof of the claim.

Now we are ready to introduce some canonical relations in the deltacategory $\mathfrak{T}_G(X)$. Let $K^{(0)} = K^{(1)} = 0$ and, for each $n \geq 2$,

$$K^{(n)} = \bigcap_{i=1}^{n-1} \mathsf{Ker}(b_i^{(n)} : X^{\otimes n} \to X^{\otimes n}) \text{ and } K = \bigoplus_{n=0}^{\infty} K^{(n)} \subseteq T(X) \ .$$

Then $K \otimes kG$ is a kG-kG-subbicomodule of $A = T(X) \otimes kG$, because each $K^{(n)}$ is a left kG-subcomodule of $X^{\otimes n}$, since each $b_i^{(n)} : X^{\otimes n} \to X^{\otimes n}$ is a morphism of left Yetter-Drinfeld kG-modules, and hence, in particular, one of left kG-comodules. It follows immediately from the above expression for Δ that $K \otimes kG$ is a coideal of $\mathfrak{T}_G(X)$. In fact, if $x \in K^{(n)}$ and $g, h \in G$ then

$$\Delta_1\Big((x \otimes gh) \otimes (g \otimes h)\Big) = \sum_{i=0}^{n} \Big((b_i^{(n)} x)^{(i)} \otimes g\Big) \otimes \Big(g^{-1} \cdot (b_i^{(n)} x)^{(i)'} \otimes h\Big)$$

$$= (1 \otimes g) \otimes (g^{-1} \cdot x \otimes h) + (x \otimes g) \otimes (1 \otimes h) \in A \otimes (K \otimes kG) + (K \otimes kG) \otimes A,$$

(K is invariant under G because each $b_i^{(n)}$ is a morphism of kG-modules); also, by definition of ϵ (and since it preserves compositions) we have $\epsilon_1\Big((X^{\otimes n} \otimes kG) \otimes^{kG} k\Big) = 0$ $\forall\ n \geq 1$, and since $K^{(0)} = 0$, we have $\epsilon_1\Big((K \otimes kG) \otimes^{kG} k\Big) = 0$.

Let J be the ideal of $\mathfrak{T}_G(X)$ generated by $K \otimes kG$. By lemma 9.4.1, J is a biideal. It is nice, in the sense of section 9.4, because over a group-like coalgebra every comodule is flat (appendix A). Therefore the quotient category

$$\mathfrak{U}_G^{+}(X) := \mathfrak{T}_G(X)/J$$

is defined, and carries a natural structure of deltacategory, by proposition 9.4.5. We call it the *binomial deltacategory* associated to the left Yetter-Drinfeld kG-module X.

9.8.3 Examples

1. The simplest non-trivial example is obtained when $G = \{1\}$ and $X = kx$

is a one-dimensional k-space. In this case each $X^{\otimes n}$ is also one-dimensional and $b_i^{(n)}$ acts on it by multiplication by $\binom{n}{i}$ (this is the case $q = 1$ of the action considered throughout appendix B). Therefore, if $\mathsf{char}k = 0$, there are no relations and $\mathfrak{U}_G^+(X)$ is just the deltacategory $\widehat{\mathrm{H}}$ corresponding to the bialgebra $H = k[x]$ of polyonomials in one variable, where

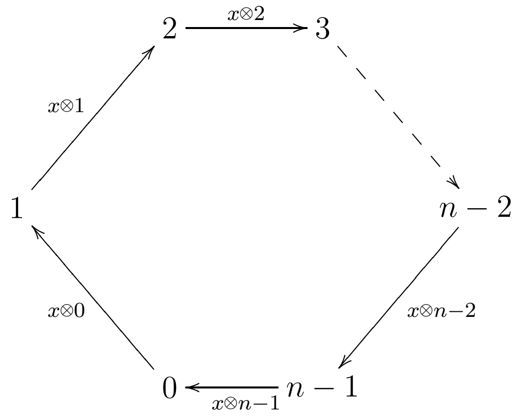
$$\Delta(x^n) = \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i} \ .$$

This is called the *binomial bialgebra*. This example explains how one may see the binomial deltacategory $\mathfrak{U}_G^+(X)$ as a generalization of the binomial bialgebra $k[x]$.

More generally, if $G = \{1\}$ and X is any k-space, then $\mathfrak{U}_G^+(X)$ is the deltacategory $\widehat{\mathrm{H}}$ corresponding to the bialgebra $H = S(X)$ (the symmetric algebra on X).

2. Let $G = \mathbb{Z}_n$ and $X = kx$ a one-dimensional k-space, viewed as Yetter-Drinfeld $k\mathbb{Z}_n$-module via $|x| = 1 \in \mathbb{Z}_n$ and $1 \cdot x = q^{-1}x \in X$, where $q \in k$ is a fixed root of unity of order n. Notice that in this case $\mathfrak{G}_G(X)$ is the linearization

of the following graph in ***Sets***



.

Again each $X^{\otimes m}$ is one-dimensional, but $b_i^{(m)}$ now acts by multiplication by the q-binomial coefficient $\left[\begin{smallmatrix} m \\ i \end{smallmatrix}\right]$, since the action of $s_1^{(2)} = \beta_{1,1}$ is

$$X \otimes X \to X \otimes X, \ \ x \otimes x \mapsto x \otimes |x|^{-1} \cdot x = q x \otimes x,$$

(see appendix B for the explanation of the relation between the actions of $s_1^{(2)}$ and $b_i^{(m)}$). Since $\left[\begin{smallmatrix} m \\ i \end{smallmatrix}\right] = 0$ for $i = 1, \ldots, m-1$ if and only if $q^m = 1$, it follows that

$$K^{(m)} = \begin{cases} X^{\otimes m} & \text{if } n \mid m, \\ \\ 0 & \text{otherwise,} \end{cases}$$

from where we see that the relations defining $\mathfrak{U}_G^+(X)$ are generated by the relations $x^{\otimes n} \otimes i \in K^{(n)} \otimes k\mathbb{Z}_n$ for $i \in \mathbb{Z}_n$. Therefore, $\mathfrak{U}_G^+(X)$ coincides with the category $\mathfrak{T}_n(q)$ of section 9.3, under

$$x \otimes i \mapsto d_i, \ \ 1 \otimes i \mapsto e_i \ .$$

Moreover, the deltacategory structures agree as well, since the deltacategory structure on $\mathfrak{U}^+_G(X)$ is given by

$$\Delta_1(x \otimes (i+j) \otimes i \otimes j) = (1 \otimes i) \otimes ((-i) \cdot x \otimes j) + (x \otimes i) \otimes (1 \otimes j)$$

$$= q^i (1 \otimes i) \otimes (x \otimes j) + (x \otimes i) \otimes (1 \otimes j) \ .$$

In particular, $\Gamma(\mathfrak{U}^+_G(X)) = T_n(q)$, Taft's Hopf algebra.

3. As before, let $G = \mathbb{Z}_n$ and $X = kx$, but now viewed as Yetter-Drinfeld $k\mathbb{Z}_n$-module via $|x| = 2$ and $1 \cdot x = q^{-1}x$, where q is a fixed root of unity of order n. Let $e \in \mathbb{Z}^+$ be defined by

$$e = \begin{cases} n & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

In this case $b_i^{(m)}$ acts by multiplication by the q^2-binomial coefficient $\left[{m \atop i} \right]$, from where it follows that the relations defining $\mathfrak{U}^+_G(X)$ are generated by $x^{\otimes e} \otimes i \in K^{(e)} \otimes k\mathbb{Z}_n$ for $i \in \mathbb{Z}_n$. Along the same lines as for Taft's Hopf algebra, it follows easily that $\Gamma(\mathfrak{U}^+_G(X))$ is generated by the admissible sections K and E, $K(i) = q^i(1 \otimes i)$ and $E(i) = x \otimes i$, subject only to the relations

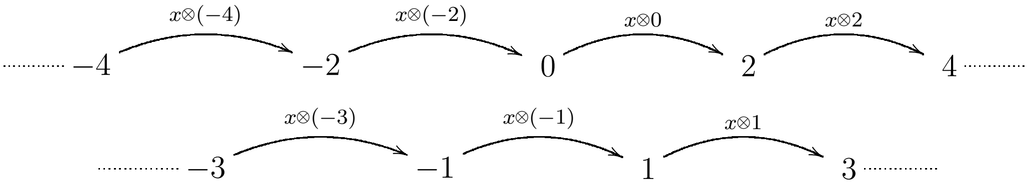
$$KE = q^2 EK \quad K^n = 1 \quad \text{and} \quad E^e = 0 \ .$$

Since $\mathfrak{U}^+_G(X))$ is a finite deltacategory, $\Gamma(\mathfrak{U}^+_G(X))$ is a bialgebra, and one computes

$$\Gamma(\Delta)(K) = K \otimes K \quad \text{and} \quad \Gamma(\Delta)(E) = 1 \otimes E + E \otimes K \ .$$

Thus, $\mathfrak{U}_G^+(X) = \bar{U}_q^+(sl_2)$, the finite-dimensional quotient of $U_q^+(sl_2)$ introduced by Lusztig.

4. Let $G = \mathbb{Z}$ and $X = kx$ a one-dimensional k-space, viewed as Yetter-Drinfeld $k\mathbb{Z}$-module via $|x| = 2 \in \mathbb{Z}$ and $1 \cdot x = q^{-1}x \in X$, where $q \in k^*$ is a fixed scalar. Notice that in this case $\mathfrak{G}_G(X)$ is the linearization of the following graph in ***Sets***



Each $X^{\otimes n}$ is one-dimensional, and $b_i^{(n)}$ acts by multiplication by the q^2-binomial coefficient $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]$. It follows that if q is not a root of unity then there are no relations, and $\mathfrak{U}_G^+(X) = \mathfrak{T}_G^+(X) = (k[x]\otimes k\mathbb{Z}, k\mathbb{Z}, \dots)$. Again, the algebra of admissible sections $\Gamma(\mathfrak{U}_G^+(X))$ contains two canonical sections

$$K : k\mathbb{Z} \longrightarrow k[x]\otimes k\mathbb{Z},\ i \mapsto q^i(1\otimes i) \text{ and } E : k\mathbb{Z} \longrightarrow k[x]\otimes k\mathbb{Z},\ i \mapsto x\otimes i\ \forall\ i \in \mathbb{Z}\ .$$

As in section 9.5 or as in the above example, one checks immediately that

$$KE = q^2 EK\ , \Gamma(\Delta)(K) = K\otimes K \ \text{ and }\ \Gamma(\Delta)(E) = 1\otimes E + E\otimes K\ .$$

This means that there is an epimorphism of bialgebras from $U_q^+(sl_2)$ onto the subalgebra of $\Gamma(\mathfrak{U}_G^+(X))$ generated by K and E. Moreover, we claim that this map is an isomorphism. To see this, recall from section 9.7.1 that

$$\Gamma(\mathfrak{U}_G^+(X)) = \mathsf{Hom}_{k\mathbb{Z}}(k\mathbb{Z}, k[x]\otimes k\mathbb{Z}) \cong \mathsf{Hom}_k(k\mathbb{Z}, k[x])$$

contains the smash product $k[x]\#k^{\mathbb{Z}}$ as a subalgebra, since after all $\mathfrak{U}^+_G(X) = \mathfrak{T}^+_G(X)$ is the category $\underset{\smile}{\mathrm{B}} \rtimes \widehat{\mathrm{H}}$ for $B = k[x]$ and $H = k\mathbb{Z}$. On the other hand, consider the morphism of algebras $\theta : k\mathbb{Z} \to k^{\mathbb{Z}}$ that sends the generator $1 \in \mathbb{Z}$ to the function $f : \mathbb{Z} \to k,\ i \mapsto q^i$. The linear extension $f : k\mathbb{Z} \to k$ is a morphism of algebras, hence it belongs to the finite dual $(k\mathbb{Z})^\circ$, and moreover, it is a group-like of this bialgebra [Mon, 1.3.5]. The powers of f are all distinct group-likes, because q is not a root of unity; hence, they are linearly independent [Swe, proposition 3.2.1]. It follows that θ maps into $(k\mathbb{Z})^\circ$, and that $\theta : k\mathbb{Z} \to (k\mathbb{Z})^\circ$ is an injective morphism of Hopf algebras. This allows us to view $k[x]$ as a $k\mathbb{Z}$-module algebra by restriction via θ from its $(k\mathbb{Z})^\circ$-module algebra structure (dual to its $k\mathbb{Z}$-comodule algebra structure). Recall from section 9.7.1 that the canonical inclusion

$$k[x]\#(k\mathbb{Z})^\circ \hookrightarrow \Gamma(\mathfrak{U}^+_G(X))$$

is a morphism of algebras. Thus, we have a morphism of algebras

$$k[x]\#k\mathbb{Z} \hookrightarrow k[x]\#(k\mathbb{Z})^\circ \hookrightarrow \Gamma(\mathfrak{U}^+_G(X)) \ .$$

One checks immediately that this composite sends x to E and $1 \in \mathbb{Z}$ to K. Since it is well-known that this map is an isomorphism

$$k[x]\#k\mathbb{Z} \cong U^+_q(sl_2),$$

the claim is proved.

The considerations of this example will be treated in greater generality later; we have included them here for motivation.

9.8.4 Binomial bialgebras

Let G be a group, X a left Yetter-Drinfeld kG-module and $\mathfrak{U}_G^+(X)$ the corresponding binomial deltacategory. If G is finite, then

$$U_G^+(X) := \Gamma(\mathfrak{U}_G^+(X))$$

is a k-bialgebra, by corollary 9.2.2. We call it the *binomial bialgebra* associated to G and X. In view of the examples of section 9.8.3, these include symmetric algebras, Taft's Hopf algebra and $\bar{U}_q^+(sl_2)$.

If G is infinite, $\Gamma(\mathfrak{U}_G^+(X))$ fails to be a bialgebra, but there may be canonical subalgebras which become bialgebras under the restriction of

$$\Gamma(\Delta) : \Gamma(\mathfrak{U}_G^+(X)) \to \Gamma\Big(\mathfrak{U}_G^+(X) \otimes \mathfrak{U}_G^+(X)\Big) \ .$$

For instance, there is such a canonical choice associated to a *bicharacter*

$$\theta : G \times G \to k^* \ .$$

Such a map is supposed to verify the following two conditions:

(1) $\theta(gh, k) = \theta(g, k)\theta(h, k)$ and

(2) $\theta(g, hk) = \theta(g, h)\theta(g, k) \ \forall \ g, h, k \in G$.

For example, if $G = \mathbb{Z}$, a bicharacter is necessarily of the form $\theta(i, j) = q^{ij}$ for some fixed $q \in k^*$.

A bicharacter yields, by (1), a morphism of algebras $kG \to (kG)^*$, $g \mapsto \theta_g = \theta(g, -)$; moreover, if μ denotes the multiplication $kG \otimes kG \to kG$, then $\mu^*(\theta_g) =$

$\theta_g\otimes\theta_g \in (kG)^*\otimes(kG)^*$ by (2). Thus, $\theta_g \in (kG)^\circ$ by [Mon, 9.1.1], and we have a morphism of Hopf algebras

$$kG \longrightarrow (kG)^\circ,\ g \mapsto \theta_g = \theta(g, -)\ .$$

This can be used to define a bialgebra of admissible sections as follows. Recall from section 9.8.2 that

$$\mathfrak{U}^+_G(X) = \mathfrak{T}_G(X)/J = (A/J, kG, \dots)$$

where $A = T(X)\otimes kG$ and J is the ideal of $T_G(X)$ generated by $K\otimes kG$, that is,

$$J = m_3\Big((T(X)\otimes kG)\otimes^{kG}(K\otimes kG)\otimes^{kG}(T(X)\otimes kG)\Big)\ .$$

It follows from the definition of composition m in $\mathfrak{T}_G(X)$ that $J = F\otimes kG$, where F is the ideal of the algebra $T(X)$ generated by K. Recall also that the category $\mathfrak{T}_G(X)$ is of the form $\underset{\vee}{\mathrm{T}}\rtimes\widehat{\mathrm{H}}$ for $T = T(X)$, $H = kG$. It follows that the quotient $\mathfrak{U}^+_G(X)$ is of this form too, for $T = T(X)/F$ and $H = kG$. Therefore (section 9.7.1), the subalgebra $(T(X)/F)\#(kG)^\circ$ of $\Gamma(\mathfrak{U}^+_G(X))$ becomes a bialgebra under the restriction of $\Gamma(\Delta)$. However, this is not yet the bialgebra we are interested in defining, as suggested by the case of $U^+_q(sl_2)$ (example 4 in section 9.8.3). Instead, we define a bialgebra $U^+_{G,\theta}(X)$ as the image of the map

$$(T(X)/F)\#kG \xrightarrow{\mathsf{id}\otimes\theta} (T(X)/F)\#(kG)^\circ\ .$$

We call $U^+_{G,\theta}(X)$ the binomial bialgebra associated to G, θ and X.

Notice that when $G = \mathbb{Z}$, $\theta(i, j) = q^{ij}$ and X is the Yetter-Drinfeld $k\mathbb{Z}$-module of example 4 in section 9.8.3, this construction reproduces the one given there, and thus we have

$$U^+_{G,\theta}(X) \cong U^+_q(sl_2)$$

when q is not a root of unity. More generally, the quantum groups $U^+_q(g)$ of Drinfeld and Jimbo will also be obtained through this construction, see section 9.8.5.

The construction of binomial deltacategories and binomial bialgebras can in fact be carried out for any Hopf algebra H (with bijective antipode) in place of kG. We now briefly describe this more general setting, pointing out the relevant differences with the case already discussed when they arise.

Let H be a Hopf k-algebra with bijective antipode λ and (X, p, χ) a left Yetter-Drinfeld H-module, as in section 9.6. Write $p(x) = x_{-1} \otimes x_0 \in H \otimes X$ and $\chi(h \otimes x) = h \cdot x \in H$. As before, the category of left Yetter-Drinfeld H-modules is braided monoidal, with braiding

$$\beta_{Y,X} : Y \otimes X \to X \otimes Y,\ \beta_{Y,X}(y \otimes x) = x_0 \otimes \lambda^{-1} x_{-1} \cdot y\ .$$

Hence, B_n acts on the tensor powers $X^{\otimes n}$ by morphisms of Yetter-Drinfeld modules.

First of all, one defines a graph in $\mathtt{Vec}_k$ $\mathfrak{G}_H(X) = (X \otimes H, H, s, t)$, with

$$s : X \otimes H \to (X \otimes H) \otimes H,\ x \otimes h \mapsto x \otimes h_1 \otimes h_2,\ t : X \otimes H \to H \otimes (X \otimes H),\ x \otimes h \mapsto x_{-1} h_1 \otimes x_0 \otimes h_2.$$

Let $\mathfrak{T}_H(X) = \mathcal{T}(\mathfrak{G}_H(X))$ be the free category on this graph, as in section 9.4. As before, $\mathfrak{T}_H(X) = (A, H, \dots)$ where $A = T(X) \otimes H$ and $T(X)$ is the tensor k-algebra of X. Again $T(X)$ inherites a structure of left H-comodule from X, and as such becomes a left H-comodule k-algebra. Moreover, $\mathfrak{T}_H(X) = \underset{\vee}{\mathrm{T}} \rtimes \widehat{\mathrm{H}}$ for $T = T(X)$ as in section 9.7.1.

In order to endow $\mathfrak{T}_H(X)$ with a deltacategory structure, we first define cofunctors of graphs $\Delta : \mathfrak{G}_H(X) \to \mathfrak{T}_H(X) \otimes \mathfrak{T}_H(X)$ and $\epsilon : \mathfrak{G}_H(X) \to \mathfrak{I}$ by

$$\begin{array}{rcl} \Delta_0 : H \otimes H & \to & H \\ h \otimes k & \mapsto & hk \end{array} \quad \text{and} \quad \begin{array}{rcl} \Delta_1 : (X \otimes H) \otimes^H (H \otimes H) & \to & A \otimes A \\ (x \otimes h_1 k_1) \otimes (h_2 \otimes k_2) & \mapsto & (1 \otimes h_2) \otimes (\lambda^{-1} h_1 \cdot x \otimes k) + \\ & & +(x \otimes h) \otimes (1 \otimes k) \end{array}$$

and

$$\begin{array}{rcl} \epsilon_0 : k & \to & H \\ 1 & \mapsto & 1 \end{array} \quad \text{and} \quad \begin{array}{rcl} \epsilon_1 : (X \otimes H) \otimes^H k & \to & k \\ (x \otimes 1) \otimes 1 & \mapsto & 0 \end{array} .$$

(In the definition of Δ_1 we have used the fact that

$$X \otimes H \otimes H \to (X \otimes H) \otimes^H (H \otimes H), \ x \otimes h \otimes k \mapsto (x \otimes h_1 k_1) \otimes (h_2 \otimes k_2)$$

is an isomorphism, which holds by definition of s and proposition 2.2.3).

These extend uniquely to cofunctors $\Delta : \mathfrak{T}_H(X) \to \mathfrak{T}_H(X) \otimes \mathfrak{T}_H(X)$ and $\epsilon : \mathfrak{T}_H(X) \to \mathfrak{I}$, which turn out to be coassociative and counital as before. Explicitly, Δ is given by

$$\Delta_1\Big((x \otimes h_1 k_1) \otimes (h_2 \otimes k_2)\Big) =$$

$$= \sum_{i=0}^{n} \Big((b_i^{(n)} x)^{(i)} \otimes h_2\Big) \otimes \Big(\lambda^{-1} h_1 \cdot (b_i^{(n)} x)^{(i)'} \otimes k\Big) \in \bigoplus_{i=0}^{n} (X^{\otimes i} \otimes H) \otimes (X^{\otimes (n-i)} \otimes H) \subseteq A \otimes A .$$

Thus, letting as before

$$K^{(n)} = \bigcap_{i=1}^{n-1} \mathsf{Ker}(b_i^{(n)} : X^{\otimes n} \to X^{\otimes n}) \text{ and } K = \bigoplus_{n=0}^{\infty} K^{(n)} \subseteq T(X) ,$$

we have that $K \otimes H$ is a a coideal of $\mathfrak{T}_H(X)$. Let F be the ideal of $T(X)$ generated by K and J the ideal of $\mathfrak{T}_H(X)$ generated by $K \otimes H$. By lemma 9.4.1, J is a biideal. Moreoever, by the same reason as before, $J = F \otimes H$. But since $\mathfrak{T}_H(X)$ is not a linear category, we have to worry about flatness: we claim that J is a nice biideal. In fact, by definition of s, $A = T(X) \otimes H$ and $J = F \otimes H$ are free as right H-comodules, hence flat (section A.1); also, $A/J \cong \Big(T(X)/F\Big) \otimes H$, which is flat as left H-comodule by one of the examples A.1.1 (here we use again that λ is bijective). By the same reason, $H \otimes H$ is flat as left H-comodule. Thus the quotient category

$$\mathfrak{U}_H^+(X) := \mathfrak{T}_H(X)/J$$

is defined, and carries a natural structure of deltacategory by proposition 9.4.5. We call it the *binomial deltacategory* associated to the left Yetter-Drinfeld H-module X.

The definition of binomial bialgebras can also be extended to this context. If the Hopf algebra H is a finite-dimensional Hopf algebra, then $\mathfrak{U}_H^+(X)$ is a finite deltacategory, so by corollary 9.2.2

$$U_H^+(X) := \Gamma(\mathfrak{U}_H^+(X))$$

is a k-bialgebra, which we call the binomial bialgebra associated to H and X (notice that in this case the antipode of H is necessarily bijective, by [Mon, 2.1.3]).

If H is infinite-dimensional (with bijective antipode), we can still construct a bialgebra of admissible sections provided that a bicharacter

$$\theta : H \otimes H \to k^*$$

is given. Such a map is supposed to verify the following two conditions:

(1) $\theta(hk \otimes l) = \theta(h, l_2)\theta(k, l_1)$ and

(2) $\theta(h \otimes kl) = \theta(h_1 \otimes k)\theta(h_2 \otimes l)\ \forall\ h, k, l \in H$.

(In particular, a *coquasitriangular* Hopf algebra is, by definition, equipped with such a bicharacter [Mon, 10.2.1]). For the case of group algebras $H = kG$, this recovers the previous notion.

A bicharacter yields, as before, a morphism of Hopf algebras

$$H \to H^{\circ op},\ h \mapsto \theta\big(h \otimes (-)\big)\ ,$$

and we may define a bialgebra $U^+_{H,\theta}(X)$ as the image of the map

$$(T(X)/F)\#H \xrightarrow{\mathsf{id}\otimes\theta} (T(X)/F)\#H^{\circ op} \hookrightarrow \Gamma(\mathfrak{U}^+_H(X))\ ,$$

which we call the binomial bialgebra associated to H, θ and X.

9.8.5 $U_q^+(g)$ as admissible sections

In this section we associate a binomial deltacategory and bialgebra to any integer square matrix A. Drinfeld and Jimbo's quantized enveloping algebra associated to a symmetrizable generalized Cartan matrix C arises from this construction by choosing A as the symmetrization of C, as will be explained.

Let $A = [a_{hk}] \in M_r(\mathbb{Z})$ be an integer square matrix of size r and $q \in k^*$ a fixed scalar, not a root of unity. Assume that $\mathsf{char}k = 0$.

Let $G = \mathbb{Z}^r$, the free abelian group of rank r, and X the vector space with basis $\{x_1, \ldots, x_r\}$, viewed as left Yetter-Drinfeld kG-module with

$$|x_k| = (a_{1k}, \ldots, a_{rk}) \in \mathbb{Z}^r, \quad (n_1, \ldots, n_r) \cdot x_h = q^{-n_h} x_h \ \forall \ (n_1, \ldots, n_r) \in \mathbb{Z}^r \ .$$

To this data there is associated, by the constructions of section 9.8.2, a binomial deltacategory

$$\mathfrak{U}_q^+(A) := \mathfrak{U}_G^+(X) \ .$$

Consider the bicharacter

$$\theta : \mathbb{Z}^r \times \mathbb{Z}^r \to k^*, \ \Big((n_1, \ldots, n_r), \ (m_1, \ldots, m_r)\Big) \mapsto q^{\sum_{i=1}^r n_i m_i} \ .$$

By means of the constructions of section 9.8.4, we may define a binomial bialgebra of admissible sections as

$$U_q^+(A) = U_{G,\theta}^+(X) \hookrightarrow \Gamma(\mathfrak{U}_q^+(A)) \ .$$

If $A = [2] \in M_1(\mathbb{Z})$, then X and θ coincide with those of example 4 in section 9.8.3; therefore, $U_q^+(A) = U_q^+(sl_2)$. More generally, we will show below that if A is the symmetrization of symmetrizable generalized Cartan matrix C, then $U_q^+(A) = U_q^+(g(C))$, the quantum enveloping algebra associated to the Kac-Moody Lie algebra defined by C.

According to the construction of section 9.8.4, $U_q^+(A) = \Big(T(X)/F\Big)\#k\mathbb{Z}^r$, where F is the ideal of $T(X)$ generated by $K = \oplus_{n=2}^{\infty} K^{(n)}$, and

$$K^{(n)} = \bigcap_{i=1}^{n-1} \mathsf{Ker}(b_i^{(n)} : X^{\otimes n} \to X^{\otimes n}) \ .$$

The explicit description of the relations F seems to be a hard problem. We are able to obtain a complete answer only for the case of Cartan matrices, and this is based on a non-trivial result of Lusztig (the description of $U_q^+(g(C))$ by means of "abstract" quantum Serre relations).

We approach this problem now. We will derive a few results about the the actions of the binomial braids $b_i^{(n)}$ on the tensor powers $X^{\otimes n}$. Some of them hold in greater generality (as will be clear from the proofs), but for simplicity we restrict from the beginning to the case where X is defined from A as above. At the end, we will specialize even further to the case of Cartan matrices. Until then, deltacategories and bialgebras will stand aside from the discussion.

Recall that the braid group B_n acts on $X^{\otimes n}$ by morphisms of left Yetter-Drinfeld kG-modules, and in such a way that $\forall\ x \in X^{\otimes n}$, $y \in X^{\otimes m}$, $s \in B_n$ and $t \in B_m$,

$$(s \otimes t)(x \otimes y) = (sx) \otimes (ty) \quad \text{and} \quad \beta_{m,n}(y \otimes x) = x \otimes |x|^{-1} \cdot y \ .$$

The action of $s_1^{(2)} = \beta_{1,1} \in B_2$ on $X \otimes X$ is then given in this case by

$$x_h \otimes x_k \mapsto q^{a_{hk}} x_k \otimes x_h \ .$$

It follows that the action of $s_i^{(n)} = 1^{(i-1)} \otimes s_1^{(2)} \otimes 1^{(n-1)} \in B_n$ on $X^{\otimes n}$ is given by

$$x_{h_1} \otimes \ldots \otimes x_{h_n} \mapsto q^{a_{h_i,h_{i+1}}} x_{h_1} \otimes \ldots \otimes x_{h_{i+1}} \otimes x_{h_i} \otimes \ldots \otimes x_{h_n} \ ;$$

hence the action of $s^{(n)}(1,i) = s_1^{(n)} \ldots s_{i-1}^{(n)} \in B_n$ on $X^{\otimes n}$ is

$$x_{h_1} \otimes \ldots \otimes x_{h_n} \mapsto q^{\sum_{j=1}^{i-1} a_{h_j,h_i}} x_{h_i} \otimes x_{h_{i-1}} \otimes \ldots \otimes x_{h_1} \otimes x_{h_{i+1}} \otimes x_{h_{i+2}} \otimes \ldots \otimes x_{h_n} \ .$$

Recall the definition of the Möbius braid $\mu^{(n)} \in kB_n$ from section B.6.2. The first observation, which is independent of the particular form of the action of B_n, is:

Lemma 9.8.1. $K^{(n)} \subseteq \mathit{Ker}(\mu^{(n)} + 1)$.

Proof. This is an immediate consequence of Cauchy's identity 26 in appendix B

$$\sum_{k=0}^{n} \mu^{(k)} \otimes 1^{(n-k)} \cdot b_k^{(n)} = 0 \ ,$$

since $b_0^{(n)} = b_n^{(n)} = 1$. □

From appendix B we know that, on the one-dimensional subspace $k\{x_i^{\otimes n}\}$ of $X^{\otimes n}$, $b_i^{(n)}$ acts by multiplication by the binomial coefficient $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_{q^{a_{ii}}}$. Since q is not a root of unity, this element is non-zero.

For fixed i and j, we will consider in particular the $(n+1)$-dimensional subspace X_{ij}^{n+1} of $X^{\otimes(n+1)}$ spanned by $x_i^{\otimes n} \otimes x_j$ and its permutations, which is clearly invariant under B_n. We abbreviate $\mathbf{x}_k := x_i^{\otimes k} \otimes x_j \otimes x_i^{\otimes(n-k)} \in X_{ij}^{n+1}$. The elements $\mathbf{x}_k$ for $k = 0, 1, \ldots, n$ for a k-basis of X_{ij}^{n+1}. It follows from the above that for each $h = 1, \ldots, n+1$, the action of $s^{(n+1)}(1, h)$ on X_{ij}^{n+1} is

$$\mathbf{x}_k \mapsto \begin{cases} q^{(h-2)a_{ii} + a_{ji}} \mathbf{x}_{k+1} & \text{if } 0 \leq k \leq h-2 \\ q^{k a_{ij}} \mathbf{x}_0 & \text{if } k = h-1 \\ q^{(h-1)a_{ii}} \mathbf{x}_k & \text{if } h \leq k \leq n \ . \end{cases}$$

Let

$$K_{ij}^{(n+1)} = K^{(n+1)} \cap X_{ij}^{n+1} = \bigcap_{h=1}^{n} \mathsf{Ker}(b_h^{(n+1)} : X_{ij}^{n+1} \to X_{ij}^{n+1}) \ .$$

Lemma 9.8.2. *The action of $\mu^{(n+1)}$ on X_{ij}^{n+1} diagonalizes, as follows. For each $k = 0, \dots, n$ let*

$$\alpha_k = \binom{n}{2} a_{ii} + k a_{ij} + (n-k) a_{ji} \ .$$

If $n = 2m+1$ is odd, there are $m+1$ eigenvectors of the form

$$q^{\alpha_{n-k}/2}\mathbf{x}_k + q^{\alpha_k/2}\mathbf{x}_{n-k} \text{ with eigenvalues } q^{(\alpha_k+\alpha_{n-k})/2}, \text{ for } k = 0, \dots, m$$

and other $m+1$ eigenvectors of the form

$$q^{\alpha_{n-k}/2}\mathbf{x}_k - q^{\alpha_k/2}\mathbf{x}_{n-k} \text{ with eigenvalues } -q^{(\alpha_k+\alpha_{n-k})/2}, \text{ for } k = 0, \dots, m \ .$$

If $n = 2m$ is even, there are m eigenvectors of the form

$$q^{\alpha_{n-k}/2}\mathbf{x}_k + q^{\alpha_k/2}\mathbf{x}_{n-k} \text{ with eigenvalues } -q^{(\alpha_k+\alpha_{n-k})/2}, \text{ for } k = 0, \dots, m-1 \ ,$$

other m eigenvectors of the form

$$q^{\alpha_{n-k}/2}\mathbf{x}_k - q^{\alpha_k/2}\mathbf{x}_{n-k} \text{ with eigenvalues } q^{(\alpha_k+\alpha_{n-k})/2}, \text{ for } k = 0, \dots, m-1 \ ,$$

plus the eigenvector

$$\mathbf{x}_m \text{ with eigenvalue } -q^{\alpha_m} \ .$$

Proof. By definition, $\mu^{(n+1)} = (-1)^{n+1} s^{(n+1)}(1, n+1) s^{(n+1)}(1, n) \dots s^{(n+1)}(1, 1)$. The action of $s^{(n+1)}(1, h)$ on the basis elements $\mathbf{x}_k$ of X_{ij}^{n+1} was described above. It follows that, for each $k = 0, \dots, n$,

$$\mu^{(n+1)} \cdot \mathbf{x}_k = (-1)^{n+1} q^{\alpha_k} \mathbf{x}_{n-k} \ .$$

Therefore, each subspace spanned by $\{\mathbf{x}_k, \mathbf{x}_{n-k}\}$ is invariant under $\mu^{(n+1)}$. If $n = 2m$ and $k = m$, this space is one-dimensional, spanned by the eigenvector $\mathbf{x}_m$ with eigenvalue $-q^{\alpha_m}$. Otherwise, this subspace is two-dimensional, and it follows readily that

$$\{q^{\alpha_{n-k}/2}\mathbf{x}_k + q^{\alpha_k/2}\mathbf{x}_{n-k},\ q^{\alpha_{n-k}/2}\mathbf{x}_k - q^{\alpha_k/2}\mathbf{x}_{n-k}\}$$

form a basis of eigenvectors, with eigenvalues as indicated according to the parity of n. □

Corollary 9.8.1. *If* $(n-1)a_{ii} + a_{ij} + a_{ji} \neq 0$*, then* $K_{ij}^{(n+1)} = 0$.

Proof. By lemma 9.8.1, $K_{ij}^{(n+1)} = 0$ if -1 is not an eigenvalue of $\mu^{(n+1)}$ in X_{ij}^{n+1}. By lemma 9.8.2, -1 is an eigenvalue of $\mu^{(n+1)}$ if and only if $\alpha_k + \alpha_{n-k} = 0$ for some $k = 0, \ldots, n$. By definition of α_k this is equivalent to $(n-1)a_{ii} + a_{ij} + a_{ji} \neq 0$. □

Lemma 9.8.3. *If for each* $h = 1, \ldots, n$ *the braid* $[h] \in B_h$ *is injective on* X_{ij}^h*, then,*

$$\mathsf{Ker}\Big(b_h^{(n+1)}|_{X_{ij}^{n+1}}\Big) = K_{ij}^{(n+1)} \quad \forall\ h = 1, \ldots, n\ .$$

Proof. Recall the factorial formulas 22 and 16

$$f^{(h)}\otimes f^{(n+1-h)} \cdot b_h^{(n+1)} = f^{(n+1)} = 1^{(n)}\otimes[1] \cdot 1^{(n-1)}\otimes[2] \cdot \ldots \cdot 1\otimes[n] \cdot [n+1]\ .$$

We claim that $1\otimes[n]$ is injective on the space X_{ij}^{n+1}. In fact, this space splits as the direct sum of the one-dimensional space spanned by $\mathbf{x}_0$ and the space $k\{x_i\}\otimes X_{ij}^n$, and both of these are invariant under $1\otimes[n]$. On the first, $1\otimes[n]$ acts by multiplication by the q-analog $[n]_{q^{a_{ii}}}$, which is non-zero since q is not a root of unity, while on the second it is injective by hypothesis. Similarly, all the lower factors $1^{(i)}\otimes[n+1-i]$ are injective on X_{ij}^{n+1}, for $i=1,\dots,n$. It follows that $f^{(h)}$ and $f^{(n+1-h)}$ are injective on X_{ij}^{n+1}, for $h=1,\dots,n$, and from here that

$$\mathsf{Ker}\Big(b_h^{(n+1)}|_{X_{ij}^{n+1}}\Big) = \mathsf{Ker}\Big([n+1]|_{X_{ij}^{n+1}}\Big) \text{ for } h=1,\dots,n \ .$$

Since $K_{ij}^{(n+1)}$ is the intersection of these kernels, the result follows. $\square$

The following result is the first one that uses in an essential way the particular form of the action of B_n on $X^{\otimes n}$ in terms of the matrix A. Below, $q^{a_{ii}}$ will be abbreviated by q_i, $q^{a_{ij}}$ by q_{ij}, $[n]_i$ will denote the q_i-analog of the natural number n and $\left[{n \atop h}\right]_i$ the q_i-analog of the binomial coefficient $\binom{n}{h}$ (the definitions of these analogs can be found in appendix B). The subindices i and j remain fixed.

Proposition 9.8.1. *For each $k=0,\dots,n$, let*

$$\lambda_k = (-1)^k \left[{n \atop k}\right]_i q^{ka_{ji}+\binom{k}{2}a_{ii}} \quad \textit{and} \quad \mathbf{x} = \sum_{k=0}^{n} \lambda_k \mathbf{x}_k \ .$$

Assume that $a_{ii}\neq 0$. Then

$$\mathit{Ker}\Big(b_1^{(n+1)}|_{X_{ij}^{n+1}}\Big) = \begin{cases} k\{\mathbf{x}\} & \textit{if } -\frac{a_{ij}+a_{ji}}{a_{ii}} \in \{0,1,2,\dots,n-1\} \\ 0 & \textit{otherwise} \ . \end{cases}$$

Proof. By definition, $b_1^{(n+1)} = \sum_{h=1}^{n+1} s^{(n+1)}(1,h)$. The action of $s^{(n+1)}(1,h)$ on the basis elements $\mathbf{x}_k$ of X_{ij}^{n+1} was described above. It follows that, for each $k =$

$0, \ldots, n$,

$$\begin{aligned} b_1^{(n+1)} \cdot \mathbf{x}_k &= \Big(q^{(1-1)a_{ii}} + q^{(2-1)a_{ii}} + \ldots + q^{(k-1)a_{ii}}\Big)\mathbf{x}_k + q^{ka_{ij}}\mathbf{x}_0 + \\ &+ \Big(q^{(k+2-2)a_{ii}+a_{ji}} + q^{(k+3-2)a_{ii}+a_{ji}} + \ldots + q^{(n+1-2)a_{ii}+a_{ji}}\Big)\mathbf{x}_{k+1} \\ &= [k]_i\mathbf{x}_k + q_{ij}^k\mathbf{x}_0 + q_{ji}q_i^k[n-k]_i\mathbf{x}_{k+1} \ . \end{aligned}$$

Let $\mathbf{y} = \sum_{k=0}^{n} \mu_k$ be an element of X_{ij}^{n+1}, where $\mu_k \in k$ are arbitrary scalars. Then

$$\begin{aligned} b_1^{(n+1)} \cdot y &= \sum_{k=0}^{n} \mu_k[k]_i\mathbf{x}_k + \sum_{k=0}^{n} \mu_k q_{ij}^k \mathbf{x}_0 + \sum_{k=0}^{n} \mu_k q_{ji} q_i^k [n-k]_i \mathbf{x}_{k+1} \\ &= \sum_{k=1}^{n} \Big(\mu_k[k]_i + \mu_{k-1} q_{ji} q_i^{k-1}[n-k+1]_i\Big)\mathbf{x}_k + \Big(\sum_{k=0}^{n} \mu_k q_{ij}^k\Big)\mathbf{x}_0 \ . \end{aligned}$$

It follows that $\mathbf{y} \in \mathsf{Ker}\Big(b_1^{(n+1)}|_{X_{ij}^{n+1}}\Big)$ if and only if

$$0 = \mu_k[k]_i + \mu_{k-1}q_{ji}q_i^{k-1}[n-k+1]_i \text{ for each } k = 1, \ldots, n \text{ and} \tag{1}$$

$$0 = \sum_{k=0}^{n} \mu_k q_{ij}^k \ . \tag{2}$$

Equation (1) determines μ_k in terms of μ_0, for $k = 1, \ldots, n$:

$$\begin{aligned} \mu_1 &= -\mu_0 q_{ji}[n]_i \\ \mu_2 &= -\mu_1 q_{ji} q_i \frac{[n-1]_i}{[2]_i} = \mu_0 q_{ji}^2 q_i \frac{[n]_i[n-1]_i}{[2]_i} = \mu_0 q_{ji}^2 q_i \left[{n \atop 2} \right]_i \\ \mu_3 &= -\mu_2 q_{ji} q_i^2 \frac{[n-2]_i}{[3]_i} = -\mu_0 q_{ji}^3 q_i^3 \frac{\left[{n \atop 2} \right]_i [n-2]_i}{[3]_i} = -\mu_0 q_{ji}^3 q_i^3 \left[{n \atop 3} \right]_i \end{aligned}$$

and in general, for $k = 1, \ldots, n$

$$\mu_k = (-1)^k \mu_0 q_{ji}^k q_i^{\binom{k}{2}} \left[{n \atop k} \right]_i = \mu_0 \lambda_k \ .$$

Thus, there are two possibilities for the kernel. If the λ_k satisfy equation (2) in place of μ_k, then the kernel is one-dimensional spanned by $\mathbf{x}$, if not, the kernel is zero. So it only remains to show that equation (2) holds for λ_k if and only if

$$-a \in \{0, 1, 2, \ldots, n-1\} \text{ where } a = \frac{a_{ij} + a_{ji}}{a_{ii}} .$$

To this end, consider the polynomial

$$f(x) = \sum_{k=0}^{n} (-1)^k q_i^{\binom{k}{2}} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_i x^{n-k} .$$

By one of Cauchy's identities for q-binomials (section B.6.2), $f(x)$ factors as follows:

$$f(x) = (x-1)(x-q_i)\ldots(x-q_i^{n-1}) . \tag{*}$$

Now, when we subsitute μ_k for λ_k in the right hand side of (2) we get

$$\begin{aligned}\sum_{k=0}^{n} q_{ij}^k \lambda_k &= \sum_{k=0}^{n} q_{ij}^k (-1)^k q_{ji}^k q_i^{\binom{k}{2}} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_i \\ &= \sum_{k=0}^{n} q_i^{ak} (-1)^k q_i^{\binom{k}{2}} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_i = \frac{f(x)}{x^n}\Big|_{x=q_i^{-a}} \\ &\overset{(*)}{=} \frac{(q^{-a}-1)(q^{-a}-q_i)\ldots(q^{-a}-q_i^{n-1})}{q^{-an}} ,\end{aligned}$$

which is zero if and only if $-a \in \{0, 1, 2, \ldots, n-1\}$. ☐

Corollary 9.8.2. *Assume that $a_{ii} \neq 0$ and let $\mathbf{x}$ be as before. Then*

$$K_{ij}^{(n+1)} = \begin{cases} k\{\mathbf{x}\} & \text{if } -\frac{a_{ij}+a_{ji}}{a_{ii}} = n-1 \\ 0 & \text{otherwise} . \end{cases}$$

Proof. If $-\frac{a_{ij}+a_{ji}}{a_{ii}} \neq n-1$ then $K_{ij}^{(n+1)} = 0$ by corollary 9.8.1. Suppose that $-\frac{a_{ij}+a_{ji}}{a_{ii}} = n-1$. Then, in particular,

$$-\frac{a_{ij}+a_{ji}}{a_{ii}} \notin \{0, 1, \ldots, k-2\} \ \forall\ k = 1, \ldots, n\ .$$

Therefore, by proposition 9.8.1, $b_1^{(k)}$ is injective on X_{ij}^k for $k = 1, \ldots, n$. But then lemma 9.8.3 applies, to conclude in particular that

$$K_{ij}^{(n+1)} = \mathsf{Ker}\Big(b_1^{(n+1)}|_{X_{ij}^{n+1}}\Big) = k\{\mathbf{x}\}\ ;$$

the last equality by proposition 9.8.1 again. □

Recall that F denotes the ideal generated by K in $T(X)$. Let

$$\bar{F}^{(n)} = \mathsf{Ker}(f^{(n)} : X^{\otimes n} \to X^{\otimes n}) \ \text{ and } \bar{F} = \bigoplus_{n=0}^{\infty} \bar{F}^{(n)}\ .$$

Our last result in this general setup is:

Lemma 9.8.4. *$\bar{F}$ is an ideal of $T(X)$, and $\bar{F} \supseteq F$.*

Proof. The factorial formula 22

$$f^{(i)} \otimes f^{(n-i)} \cdot b_i^{(n)} = f^{(n)}$$

shows that $\mathsf{Ker} b_i^{(n)} \subseteq \mathsf{Ker} f^{(n)}\ \forall\ i$; in particular, $K^{(n)} \subseteq \bar{F}^{(n)}$, so $K \subseteq \bar{F}$. On the other hand, applying the horizontal symmetry $*$ (section B.2.3) to the formula above yields, by formulas 7 and 18 in appendix B,

$$b_i^{(n)^*} \cdot (f^{(i)} \otimes f^{(n-i)}) = f^{(n)}\ .$$

This implies that $\bar{F}$ is an ideal of $T(X)$. Since it contains K, it must also contain F. □

Now we go back to consideration of the bialgebra

$$U_q^+(A) = \Big(T(X)/F\Big)\# k\mathbb{Z}^r \hookrightarrow \Gamma(\mathfrak{U}_q^+(A))$$

This algebra is generated by the following elements, for $i = 1, \dots, r$

$$E_i = x_i \otimes (0, \dots, 0) \ \text{ and } \ K_i = 1 \otimes (0, \dots, 0, 1, 0, \dots, 0) \ ,$$

(where the 1 appears in the i-th coordinate). When viewed as admissible sections, we have

$$E_i(n_1, \dots, n_r) = x_i \otimes (n_1, \dots, n_r) \ \text{ and } \ K_i(n_1, \dots, n_r) = q^{n_i} 1 \otimes (n_1, \dots, n_r) \ .$$

It follows easily from the definition of smash product, or by computing the product of admissible sections directly, that

$$K_i K_j = K_j K_i \tag{1}$$

and

$$K_i E_j = q^{a_{ij}} E_j K_i \ . \tag{2}$$

For general A, the only additional conclusion we have obtained is that

$$\sum_{k=0}^{n} (-1)^k \left[{n \atop k} \right]_i q^{k a_{ji} + \binom{k}{2} a_{ii}} E_i^k E_j E_i^{n-k} = 0 \qquad \text{whenever } a_{ii} \neq 0 \text{ and } (n-1)a_{ii} + a_{ij} + a_{ji} = 0 \tag{3}$$

In fact, this is just a reformulation of corollary 9.8.2.

In general, there is no reason why these relations should generate all relations F in $U_q^+(A)$. However, this turns out to be the case in the special case of Cartan matrices, that we now consider.

Let $C = [c_{ij}] \in M_r(\mathbb{Z})$ be a *generalized Cartan matrix.* This means that

$$c_{ii} = 2 \ \forall \ i = 1, \ldots , r$$

$$c_{ij} \leq 0 \text{ for } i \neq j$$

$$\text{if } c_{ij} = 0 \text{ then } c_{ji} = 0 \ .$$

Suppose in addition that C is *symmetrizable.* This means that there is an invertible diagonal matrix $D \in M_r(\mathbb{Z})$ such that DC is symmetric. In this case, D is unique up to a constant factor, and all its entries have the same sign. The canonical symmetrization of C,

$$A := DC,$$

is the one corresponding to the choice of D with minimum positive integer entries.

A generalized Cartan matrix is of *finite type* if it is positive-definite. Such Cartan matrices are always symmetrizable. Finite-dimensional semisimple Lie algebras over $\mathbb{C}$ are in one-to-one correspondence with symmetrizable generalized Cartan matrices of finite type. For instance, the Cartan matrix of $sl_{r+1}(\mathbb{C})$ is

$$\begin{bmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ 0 & -1 & 2 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & \ldots & 0 & -1 & 2 \end{bmatrix}$$

(a square matrix of size r). For more details on Cartan matrices the reader is referred to [Kac, chapters 1,2 and 4].

Associated to any symmetrizable generalized Cartan matrix C there is Lie algebra $g(C)$, called a Kac-Moody Lie algebra, and a quantum group (bialgebra) $U_q(g(C))$, defined by means of generators and relations [Jan, 4.3]. These were first defined by Drinfeld [Dri1] and Jimbo [Jim]. We shall concentrate on the subalgebra $U_q^+(g(C))$, which is defined by generators K_i and E_i, for $i = 1, \dots, r$ subject to the relations that each K_i be invertible,

$$K_i K_j = K_j K_i \tag{1'}$$

$$K_i E_j = q^{d_i c_{ij}} E_j K_i \tag{2'}$$

and

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}_{q^{d_i}} E_i^k E_j E_i^{n-k} = 0 \text{ whenever } c_{ij} = 1 - n \ . \tag{3'}$$

Here, c_{ij} are the entries of C, d_i are the (diagonal) entries of D and the q-binomial coefficient $\binom{n}{k}_q$ is that of [K, VI.1.6] (warning: the notation for binomials in this work is precisely the opposite of [K]). In terms of the q-binomials $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ of appendix B and the present section, these other binomials are [K, VI.1.8]

$$\binom{n}{k}_q = q^{-k(n-k)} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{q^2} \ .$$

Let us compare relations (1')-(3') with those obtained for $U_q^+(A)$ above. The entries of A, C and D are related by $a_{ij} = d_i c_{ij}$; $c_{ii} = 2$ and $a_{ij} = a_{ji}$. Thus relation (2') corresponds to (2), and obviously (1') to (1). One checks easily that

$$(n-1)a_{ii} + a_{ij} + a_{ji} = 0 \Leftrightarrow c_{ij} = 1 - n \ ,$$

and that in this case

$$(-1)^k \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{q^{a_{ii}}} q^{ka_{ji}+\binom{k}{2}a_{ii}} = (-1)^k \binom{n}{k}_{q^{d_i}} ,$$

so that (3') corresponds to (3) too. This means that there is a well-defined epimorphism of algebras

$$U_q^+(g(C)) \twoheadrightarrow U_q^+(A) .$$

One checks easily that this is also a morphism of bialgebras, where the coalgebra structure on $U_q^+(g(C))$ is

$$\Delta(K_i) = K_i{\otimes}K_i,\ \Delta(E_i) = E_i{\otimes}1 + K_i{\otimes}E_i,\ \epsilon(K_i) = 1 \text{ and } \epsilon(E_i) = 0 .$$

We claim that this is actually an isomorphism.

Recall that $U_q^+(A) = \Big(T(X)/F\Big)\#k\mathbb{Z}^r$, where F is the ideal of $T(X)$ generated by K.

Let S be the ideal of $T(X)$ generated by the *quantum Serre relations*

$$S_{ij}^n := \sum_{k=0}^{n} (-1)^k \binom{n}{k}_{q^{2d_i}} x_i^{\otimes k}{\otimes}x_j{\otimes}x_i^{\otimes(n-k)} \in T(X) ,$$

for those i, j and n for which $c_{ij} = 1 - n$.

The element S_{ij}^n is precisely the element $\mathbf{x}$ of proposition 9.8.1, after changing notation as indicated above. Therefore, by corollary 9.8.2, $S_{ij}^n \in K$. Together with lemma 9.8.4, this gives

$$S \subseteq F \subseteq \bar{F} ,$$

where $\bar{F}$ is the direct sum of the kernels of the factorial braids $f^{(n)}$ as above.

On the other hand, Lusztig's result [Lus, 33.1.5] shows that $\bar{F}$ is generated by the quantum Serre relations. [In Lusztig's book, $T(X)$ is denoted by f, and $\bar{F}$ is defined as the radical of certain bilinear form on f [Lus, 1.2.4, 3.1.1]. Schauenburg has noted that this coincides with $\bar{F}$ as defined here [Sch, 3.1].]

Therefore, $S = F = \bar{F}$. Thus, we have obtained

$$U_q^+(A) = \Big(T(X)/S\Big)\#k\mathbb{Z}^r \ .$$

But this is the well-known smash product presentation of $U_q^+(g(C))$, as in [Sch, 4.2]. Therefore, $U_q^+(A) \cong U_q^+(g(C))$ as claimed.

Summarizing: to each integer matrix A there is canonically associated a bialgebra $U_q^+(A)$, either as certain admissible sections of a deltacategory or as a certain biproduct. If A is the symmetrization of a Cartan matrix C, then $U_q^+(A) = U_q^+(g(C))$, the quantum groups of Drinfeld and Jimbo.

Chapter 10

Categories in Algebras

In this chapter we study categories internal to a category of monoids $\mathbb{S} = \mathtt{Mon}_{\mathbb{S}_o}$ in a given monoidal category $\mathbb{S}_o$. We present some results that extend well-known facts about categories in $\mathtt{Groups}$, which is the particular case $\mathbb{S}_o = \mathtt{Sets}$. We are mainly interested in the case $\mathbb{S}_o = \mathtt{Vec}_k$, $\mathbb{S} = \mathtt{Alg}_k$, but the results hold in this more general setting of arbitrary monoidal categories.

Let us describe the contents of this chapter in more detail.

It is well-known that the concept of a category in $\mathtt{Groups}$ is equivalent to that of a cat^{1}group and to that of a crossed module of groups [Lod1]. We review these equivalences in section 10.1, pointing out that they hold true for cat^{1}monoids for which the base monoid is a group. This slightly more general setting is more natural from the point of view of internal categories, as it permits generalization to the case when $\mathbb{S}_o$ is arbitrary, instead of $\mathbb{S}_o = \mathtt{Sets}$. In this section we also describe the monoid of admissible sections of a category in $\mathtt{Groups}$, and show that it coincides with Whitehead's monoid of derivations of the corresponding crossed module.

Section 10.2 deals with the generalization mentioned above. Namely, cat^1-monoids in $\mathcal{S}_o$ are defined, and the equivalence with categories in $\mathit{Mon}_{\mathcal{S}_o}$ is obtained, when the base monoid is actually a Hopf monoid. Results of this type are known, not only for the case of cat^1-groups mentioned above, but more generally for internal categories to lex categories $\mathcal{S}_o$ (as in examples 2.3.1) [CPP]. The result presented here goes one step further, in the sense that arbitrary monoidal categories $\mathcal{S}_o$ are considered, instead of only lex ones. These results are sometimes known as the *generalized Eckmann-Hilton argument*, for the reason that the one-object case of the equivalence between categories in $\mathit{Mon}_{\mathcal{S}_o}$ and cat^1-monoids in $\mathcal{S}_o$ simply says that a monoid in $\mathit{Mon}_{\mathcal{S}_o}$ is precisely a commutative monoid in $\mathcal{S}_o$.

Finally, these considerations are applied in section 10.3 to associate a cat^1-algebra to any finite-dimensional quasitriangular Hopf algebra H. A morphism of quasitriangular Hopf algebras induces a cofunctor between the corresponding categories in Alg_k. Therefore, composing with the admissible sections functor (section 5.3), one obtains a monoid (or a group, if only invertible admissible sections are considered), which is an invariant of finite-dimensional quasitriangular Hopf algebras. Further study of this invariant is not pursued in this work.

The observation that Drinfeld's double of a quasitriangular Hopf algebra was part of a structure somewhat analogous to a cat^1-group, was what first led us to consideration of the general notion of internal categories.

For lack of time and space, the results of this chapter will be stated without proofs. These will we provided in a separate work.

10.1 Cat^1-groups

Cat^1-groups were introduced by Loday [Lod1] as algebraic models of homotopy 2-types, along with analogous higher dimensional notions.

Definition 10.1.1. A cat^1-group is a diagram of groups $G \underset{t}{\overset{s}{\rightrightarrows}} N \xrightarrow{i} G$ such that $si = ti = \mathsf{id}_N$ and $[\mathsf{Ker} s, \mathsf{Ker} t] = 1$ (that is, $\mathsf{Ker} s$ and $\mathsf{Ker} t$ commute elementwise).

Crossed modules were introduced by Whitehead [W].

Definition 10.1.2. A crossed module of groups consists of a morphism of groups $\partial : K \to N$, together with a left action of N on K by automorphisms, such that

1. $\partial(n \cdot k) = n\partial(k)n^{-1}$
2. $\partial(n) \cdot m = nmn^{-1}$

The following result is well-known [Lod1, lemma 2.2].

Proposition 10.1.1. *The following data are equivalent:*

1. *a cat^1-group* $G \underset{t}{\overset{s}{\rightrightarrows}} N$ *,*
2. *a crossed module of groups* $\partial : K \to N$*,*
3. *a category in* ***Groups*** (G, N, s, t, i, m)*.*

Proof. (Sketch). The crossed module is obtained from the cat^1-group by setting $K = \mathsf{Ker} s$, letting ∂ be the restriction of t and making N act on K by restriction along $i : N \to G$ from the conjugation action of G on K. From the crossed module one obtains a category in ***Groups*** with $G = K \rtimes N$ (semidirect product with respect to the given action of N on K), and

$$\begin{array}{rclrcl} s : G & \to & N, & (k,n) & \mapsto & n \\ t : G & \to & N, & (k,n) & \mapsto & \partial(k)n \\ i : N & \to & G, & n & \mapsto & (1,n) \\ m : G \times^N G & \to & G, & (h, \partial(k)n, k, n) & \mapsto & (hk, n)\ . \end{array}$$

From the category in ***Groups*** one obtains a cat^{1}group simply by forgetting the additional structure m. ☐

The *Whitehead's monoid of derivations* $\mathsf{Der}(N,K)$ of a crossed module $\partial : K \to N$ was introduced in [W]; this terminology and notation is taken from [N]. Explicitly,

$$\mathsf{Der}(N,K) = \{D : N \to K \ / \ D(nm) = (Dn)(n \cdot Dm) \ \forall \ n,m \in N\} \ ;$$

with multiplication

$$(D_1 * D_2)(n) = D_1\Big((\partial D_2 n)n\Big)D_2(n)$$

and unit element $D_0 : N \to K$, $D_0(n) = 1 \ \forall \ n \in N$.

The following result was already announced in section 5.2. Its proof is straightforward.

Proposition 10.1.2. *The monoid of admissible sections of a category in **Groups** coincides with Whitehead's monoid of derivations of the corresponding crossed module of groups.*

The results of proposition 10.1.1 can be slightly generalized, as follows. One can define cat^{1}monoids, simply by replacing the word *group* by the word *monoid* in definition 10.1.1. One can also relax the requirements in definition 10.1.2 of crossed modules, by letting K be any monoid (while retaining that N be a group). With these conventions, we have

Proposition 10.1.3. *Let N be a group. The following data are equivalent:*

1. *a cat^1-monoid $G \overset{s}{\underset{t}{\rightrightarrows}} N$,*
2. *a crossed module of monoids $\partial : K \to N$,*
3. *a category in* ***Monoids*** *(G, N, s, t, i, m).*

This setting is more natural from the point of view of internal categories, as we explain in the next section.

10.2 Cat^1-algebras

Let $\mathcal{S}_o$ be a symmetric regular monoidal category and $\mathcal{S} = \mathit{Mon}_{\mathcal{S}_o}$ the (symmetric, regular) monoidal category of monoids in $\mathcal{S}_o$ (proposition 2.1.1). Let (A, H, s, t) be a graph in $\mathcal{S}$; according to definition 2.3.1, this means that H is a bimonoid in $\mathcal{S}_o$ and A an H-bicomodule monoid in $\mathcal{S}_o$. Let I be the unit object of $\mathcal{S}_o$ and $\mathsf{u}_H : I \to H$ and $\mathsf{u}_A : I \to A$ the unit maps of the monoids H and A. We define

$$\mathsf{K}_r(s) := A \overset{H}{\otimes} I = \mathsf{Eq}_{\mathcal{S}_o}\big(A \overset{s}{\underset{\mathsf{id}_A \otimes \mathsf{u}_H}{\rightrightarrows}} A \otimes H \big) \text{ and } \mathsf{K}_l(t) := I \overset{H}{\otimes} A = \mathsf{Eq}_{\mathcal{S}_o}\big(A \overset{t}{\underset{\mathsf{u}_H \otimes \mathsf{id}_A}{\rightrightarrows}} H \otimes A \big) .$$

When $\mathcal{S}_o = \mathit{Sets}$, s is necessarily of the form $s(a) = (a, \tilde{s}(a))$ for some map $\tilde{s} : A \to H$, and hence $\mathsf{K}_r(s) = \{a \in A \ / \ \tilde{s}(a) = 1 \ \} = \mathsf{Ker}\tilde{s}$. Similarly $\mathsf{K}_l(t) = \mathsf{Ker}\tilde{t}$.

When $\mathcal{S}_o = \mathit{Vec}_k$, $\mathsf{K}_r(s)$ and $\mathsf{K}_l(t)$ are the spaces of *right and left coinvariants* of the H-H-bicomodule A [Mon, 1.7.1.2]

Definition 10.2.1. Let $\mathcal{S}_o$ be a symmetric regular monoidal category. A cat^1-monoid in $\mathcal{S}_o$ consists of a 5-tuple (A, H, s, t, i) where

- H is a bimonoid in $\mathcal{S}_o$,
- (A, s, t) is an H-H-bicomodule monoid,
- $i : H \to A$ is a morphism of H-H-bicomodule monoids, and
- $\mathsf{K}_r(s)$ and $\mathsf{K}_l(t)$ commute inside the monoid A.

It is clear that, in the case $\mathcal{S}_o = \mathit{Sets}$, this definition recovers the notion of cat^1-monoids discussed in section 10.1. To obtain a description of categories in $\mathcal{S}$ in terms of cat^1-monoids in $\mathcal{S}_o$ we need to make some assumptions on H, that replace the assumption that H be a group in the case $\mathcal{S}_o = \mathit{Sets}$. The natural option is to assume that H is a *Hopf monoid* in $\mathcal{S}_o$. Let $\lambda_H : H \to H$ denote the antipode of H (a morphism in $\mathcal{S}_o$). Let (A, s, t) be an H-H-bicomodule and $i : H \to A$ a morphism of H-H-bicomodule monoids. Then i is convolution-invertible in the (ordinary) monoid $\mathsf{Hom}_{\mathcal{S}_o}(H, A)$ with convolution-inverse $\bar{i} = i \circ \lambda_H : H \to A$. We define the *right and left traces* of A over H with respect to i as

$$\mathsf{t}_r : A \xrightarrow{s} A{\otimes}H \xrightarrow{\mathsf{id}_A \otimes \bar{i}} A{\otimes}A \xrightarrow{\mu_A} A \text{ and } \mathsf{t}_l : A \xrightarrow{t} H{\otimes}A \xrightarrow{\bar{i} \otimes \mathsf{id}_A} A{\otimes}A \xrightarrow{\mu_A} A\ .$$

Maps of these sort have been considered in the literature for the Hopf algebra case [Rad], sometimes under the name of *total integrals* [Doi2].

Proposition 10.2.1. *Let $\mathcal{S}_0$ and $\mathcal{S}$ be as before. Let H be a Hopf monoid in $\mathcal{S}_0$. Then, given a category (A, H, s, t, i, m) in $\mathcal{S}$, (A, H, s, t, i) is a cat^1-monoid in $\mathcal{S}_0$, and conversely, any cat^1-monoid (A, H, s, t, i) in $\mathcal{S}_0$ carries a unique structure of category in $\mathcal{S}$, with composition m given by any of the following*

$$m : A{\otimes^H}A \xrightarrow{\mathsf{can}} A{\otimes}A \xrightarrow{t_r \otimes id_A} A{\otimes}A \xrightarrow{\mu_A} A \ \mathit{or}\ m : A{\otimes^H}A \xrightarrow{\mathsf{can}} A{\otimes}A \xrightarrow{id_A \otimes t_l} A{\otimes}A \xrightarrow{\mu_A} A \ .$$

It is also possible to obtain a description of cat^1-monoids in terms of *internal crossed modules*, extending the description of cat^1-groups in terms of crossed modules of groups of section 10.1. More generally, one can associate a simplicial object to any augmented internal category (a category equipped with a functor to the *one-arrow* category), called its nerve, and define, when $\mathcal{S} = \mathit{Mon}_{\mathcal{S}_0}$, a *Moore functor* from simplicial objects in $\mathcal{S}$ to complexes of monoids in $\mathcal{S}_0$, extending the corresponding theory for simplicial groups.

10.3 Drinfeld's double as a cat^1-algebra

Let H be a quasitriangular Hopf algebra with R-matrix R [Mon, 10.1.5] or [K, VIII.2.2]. It is well-known that there is a corresponding morphism of Hopf algebras

$$\varphi_R : D(H) \to H, \ \ \varphi_R(f \bowtie h) = f(R_i)R_i'h \ ,$$

where $D(H)$ is Drinfeld's double (in the *left handed* version of [K, IX.4.1], not as in section 9.6). Since $\tilde{R} = \tau(R)^{-1}$ is another R-matrix for H, there is another morphism of Hopf algebras $D(H) \xrightarrow{\varphi_{\tilde{R}}} H$. It turns out that the pair $D(H) \underset{\varphi_{\tilde{R}}}{\overset{\varphi_R}{\rightrightarrows}} H$ satisfies some properties formally similar to those defining a cat^1-group (definition

10.1.1 above). More precisely, there is a cat^1-algebra of the form $(D(H), H, \dots)$, where $D(H)$ is viewed as H-H-bicomodule algebra by corestriction along φ_R and $\varphi_{\tilde{R}}$:

Proposition 10.3.1. *Let H be a finite-dimensional quasitriangular Hopf algebra with R-matrix $R = \sum_i R_i {\otimes} R'_i$. Then there is a cat^1-monoid in $\mathcal{S}_0 = \mathbf{Vec}_k$ of the form $(D(H), H, s, t, i)$ where*

- *$D(H) = (H^*)^{cop} \bowtie H$ is Drinfeld's double of H, as above,*
- *$i : H \to D(H)$ is the canonical inclusion $i(h) = 1{\otimes}h$,*
- *$s : D(H) \to D(H){\otimes}H$ is $s(f \bowtie h) = \sum_i f_1(R_i)(f_2 \bowtie h_1){\otimes}R'_i h_2$, and*
- *$t : D(H) \to H{\otimes}D(H)$ is $t(f \bowtie h) = \sum_j f_2(S_j) S'_j h_1 {\otimes} (f_1 \bowtie h_2)$,*

where $S = \tau(R)^{-1} = \sum_j S_j {\otimes} S'_j$.

Let $\mathfrak{C}_R$ be the category in $\mathbf{Alg}_k$ corresponding to a finite-dimensional quasitriangular Hopf algebra (H, R), by means of propositions 10.3.1 and 10.2.1.

A morphism between quasitriangular Hopf algebras (H, R) and (K, S) is a morphism of Hopf algebras $\varphi : H \to K$ such that $(\varphi{\otimes}\varphi)(R) = S$. Let $\mathbf{Qth}_k$ denote the category of finite-dimensional quasitriangular Hopf k-algebras. The relevance of cofunctors is once again made clear, by the following result.

Proposition 10.3.2. *A morphism of finite-dimensional quasitriangular Hopf algebras $\varphi_0 : H \to K$ induces a cofunctor $\varphi : \mathfrak{C}_R \to \mathfrak{C}_S$. This gives a functor*

$$\mathbf{Qth}_k \to \overleftarrow{\mathbf{Cat}}_{\mathbf{Alg}_k} \ .$$

By composing with the admissible sections functor $\Gamma : \overleftarrow{\mathtt{Cat}}_{\mathtt{Alg}_k} \to \mathtt{Monoids}$ (section 5.3), we get a functor

$$\mathtt{Qth}_k \to \mathtt{Monoids}\,,$$

an invariant of finite-dimensional quasitriangular Hopf algebras.

Appendix A

On comodules over a coalgebra

In this appendix we discuss various properties of the category of comodules over a k-coalgebra that are used in the main body of this thesis. In section A.1 we recall the definition of flat comodules and other related notions. In section A.2 we collect a series of basic facts about comodules and modules over the dual algebra. Finally, in section A.3 we construct products and coproducts in the category of comodules. Most of these results are straightforward or can be found scattered through the literature. The main references are [Doi1] and [T1, appendix 2].

We will use the abbreviated Sweedler's notation: comultiplications $\Delta_C : C \to C \otimes C$ and comodule structure maps $t : M \to C \otimes M$ and $s : M \to M \otimes C$ are denoted

$$\Delta_C(c) = c_1 \otimes c_2, \ t(m) = m_{-1} \otimes m_0 \text{ and } s(m) = m_0 \otimes m_1$$

respectively. Notice that the subindex 0 in comodule structure maps is reserved for the component that belongs to M, in agreement with [Mon, 1.6.2], but summation signs are omitted whenever possible. Negative subindices encode coassociativity as follows:

$$(\mathsf{id}_C \otimes t)t(m) = m_{-2} \otimes m_{-1} \otimes m_0 = (\Delta_C \otimes \mathsf{id}_M)t(m) \ .$$

The categories of left and right C-comodules over a k-coalgebra C will be denoted respectively by $\mathsf{Comod}^l C$ and $\mathsf{Comod}^r C$. k will be a field. Hom^l_C and Hom^r_C will stand for homomorphisms of left and right C-comodules respectively.

A.1 Flatness

The tensor coproduct $M_1 \otimes^C M_2$ of a right C-comodule (M_1, s) with a left C-comodule (M_2, t) is the equalizer of the pair $M_1 \otimes M_2 \underset{\mathsf{id}_1 \otimes t}{\overset{s \otimes \mathsf{id}_2}{\rightrightarrows}} M_1 \otimes C \otimes M_2$. It was defined in the more general context of regular monoidal categories in section 2.2, where some of its basic properties were proved.

A left C-comodule M is called *flat* if the functor

$$(-) \otimes^C M : \mathsf{Comod}^r C \to \mathit{Vec}_k$$

is exact, and *injective* if the functor

$$\mathsf{Hom}^l_C(-, M) : \mathsf{Comod}^l C \to \mathit{Vec}_k$$

is exact. These two notions are actually equivalent: by [T1, proposition A.2.1], M is flat if and only if it is injective. Moreover, this is the case if and only if the functor $(-) \otimes^C M$ preserves epimorphisms. In fact, we know from remark 2.2.1 that the tensor coproduct always preserves monomorphisms.

Let V be a vector k-space. Then $C \otimes V$ is a left C-comodule with structure map $\Delta_C \otimes \mathsf{id}_V$ (in the terminology of chapter 3, this comodule is obtained from the vector space V by coinduction along ϵ_C). A left C-comodule M is called *free* if it is isomorphic to a left C-comodule $C \otimes V$ as above. By [Doi1, corollary 1], every free comodule is flat. (This is an immediate consequence of propositions 2.2.2 and 2.2.3). This result will be complemented with that of lemma A.3.4 below.

Examples A.1.1.

Let X be a set and $C = kX$ the group-like coalgebra on X: $\Delta_C(x) = x \otimes x$ and $\epsilon_C(x) = 1 \; \forall \, x \in X$. Then every left C-comodule is flat. In fact, as already mentioned in section 9.1, a left C-comodule (M, t) decomposes as

$$M = \oplus_{x \in X} M_x \text{ where } M_x = \{m \in M \; / \; t(m) = x \otimes m\},$$

and similarly for right C-comodules. Consequently, there are equivalences of categories

$$\mathsf{Comod}^r C \cong \coprod_{x \in X} \mathit{Vec}_k \cong \mathsf{Comod}^l C \ ,$$

preserving the additive structure. The functors

$$(-) \otimes^C M : \mathsf{Comod}^r C \to \mathit{Vec}_k \quad \text{and} \quad \coprod_{x \in X} (-) \otimes M_x : \coprod_{x \in X} \mathit{Vec}_k \to \mathit{Vec}_k$$

correspond to each other under these equivalences. Since the latter is clearly exact, so is the former.

Let H be a Hopf algebra. We claim that $H \otimes H$, viewed as left H-comodule via

$$H \otimes H \to H \otimes (H \otimes H), \quad h \otimes k \mapsto h_1 k_1 \otimes h_2 \otimes k_2 \ ,$$

is flat. Indeed, $H \otimes H$ is a left Hopf H-module (section 9.7.2 or [Mon, 1.9.1]) when equipped with the left H-module structure

$$H \otimes (H \otimes H) \to H \otimes H, \quad h \otimes (f \otimes g) \mapsto hf \otimes g \ .$$

Hence, by the fundamental theorem on Hopf modules (corollary 9.7.1 or [Mon, 1.9.4]), $H{\otimes}H$ is free as left H-comodule, in particular flat.

However, if H is only a bialgebra, $H{\otimes}H$ may fail to be flat as left H-comodule. To see this, consider the bialgebra

$$B = k[x]/(x^2 - x^3) \text{ with } \Delta(x) = x{\otimes}x \text{ and } \epsilon(x) = 1 \ .$$

Let $M = k\alpha$ be a one-dimensional space, turned into a (left) B-module via $x\cdot\alpha = 0$. Then M is not projective as B-module, because the surjection

$$p : B \to M,\ p(1) = x - x^2,\ p(x) = p(x^2) = 0,$$

does not split, since the only morphisms of B-modules $M \to B$ are

$$j : M \to B,\ j(\alpha) = x - x^2 \text{ and its linear multiples}$$

(since $x - x^2$ and its linear multiples are the only elements of B annihilated by x), and we have $pj = 0$. On the other hand, view $B{\otimes}B$ as B-module by restriction via Δ. Then M is a direct summand of $B{\otimes}B$, since

$$B{\otimes}B = k\{x{\otimes}1 - x^2{\otimes}1\} \oplus k\{x{\otimes}1 + x^2{\otimes}1, 1{\otimes}1, x{\otimes}x, x{\otimes}x^2, x^2{\otimes}x, x^2{\otimes}x^2, 1{\otimes}x, 1{\otimes}x^2\},$$

is a direct decomposition of $B{\otimes}B$ into B-modules (assuming $\mathsf{char} k \neq 2$), and

$$M \cong k\{x{\otimes}1 - x^2{\otimes}1\} \text{ via } \alpha \mapsto x{\otimes}1 - x^2{\otimes}1 \ .$$

Therefore, $B{\otimes}B$ is not projective as B-module. Let $H = B^*$. It follows that $H{\otimes}H = (B{\otimes}B)^*$ is not injective as B-module, or equivalently as H-comodule (by proposition 1.1.4 in [Doi1]). As already mentioned, injective=flat for comodules, so we have an example of a bialgebra H for which $H \otimes H$ is not flat as (left) H-comodule. The author thanks Warren Nichols for showing him this example.

The final example is a variant on the result above about the flatness of $H \otimes H$. Let H be a Hopf k-algebra with bijective antipode and X a left H-comodule, via $x \mapsto x_{-1} \otimes x_0$. View $X \otimes H$ as left H-comodule via

$$x \otimes h \mapsto x_{-1} h_1 \otimes x_0 \otimes h_2 \ .$$

Then $X \otimes H$ is flat as left H-comodule; in fact, it is free, by the fundamental theorem on Hopf modules in its version for left comodules and right modules (as in the remarks following 1.9.4 in [Mon]; this uses the assumption on the antipode), since it is trivial to verify that it becomes a Hopf module when equipped with the right H-module structure

$$(X \otimes H) \otimes H \to X \otimes H, \quad (x \otimes h) \otimes k \mapsto x \otimes hk \ .$$

A subspace M of a left C-comodule (A, t) is called a (left) subcomodule if $t(M) \subseteq C \otimes M$.

Lemma A.1.1. *Let C be a k-coalgebra, A_1 a right C-comodule and A_2 a left one. Let M_1 and N_1 be right C-subcomodules of A_1 and M_2 and N_2 left C-subcomodules of A_2. Then*

$$(M_1 \otimes^C M_2) \cap (N_1 \otimes^C N_2) = (M_1 \cap N_1) \otimes^C (M_2 \cap N_2)$$

as subspaces of $A_1 \otimes A_2$.

Proof. First notice that for any pair of maps $A \rightrightarrows B$ and subspace N of A, we have

$$\mathsf{Eq}_k(\ A \rightrightarrows B \ ) \cap N = \mathsf{Eq}_k(A \cap N \hookrightarrow \ A \rightrightarrows B \ ) \ .$$

Therefore, since $(M_1{\otimes}M_2)\cap(N_1{\otimes}N_2)=(M_1\cap N_1){\otimes}(M_2\cap N_2)$, we have

$$\begin{aligned}
&\mathsf{Eq}_k(\ M_1{\otimes}M_2 \rightrightarrows M_1{\otimes}C{\otimes}M_2\ )\cap(N_1{\otimes}N_2) = \\
&\qquad = \mathsf{Eq}_k\Big((M_1{\otimes}M_2)\cap(N_1{\otimes}N_2) \hookrightarrow\ M_1{\otimes}M_2 \rightrightarrows M_1{\otimes}C{\otimes}M_2\ \Big) \\
&\qquad = \mathsf{Eq}_k\Big(\ (M_1\cap N_1){\otimes}(M_2\cap N_2) \rightrightarrows (M_1\cap N_1){\otimes}C{\otimes}(M_2\cap N_2)\ \Big) \\
&\qquad\qquad = (M_1\cap N_1){\otimes}^C(M_2\cap N_2),
\end{aligned}$$

thus

$$(M_1{\otimes}^C M_2)\cap(N_1{\otimes}N_2)=(M_1\cap N_1){\otimes}^C(M_2\cap N_2).$$

Hence, by symmetry,

$$(M_1{\otimes}M_2)\cap(N_1{\otimes}^C N_2)=(M_1\cap N_1){\otimes}^C(M_2\cap N_2).$$

But then also

$$\begin{aligned}
(M_1\cap N_1){\otimes}^C(M_2\cap N_2) &= (M_1{\otimes}^C M_2)\cap(N_1{\otimes}N_2)\cap(M_1{\otimes}M_2)\cap(N_1{\otimes}^C N_2) \\
&= (M_1{\otimes}^C M_2)\cap(N_1{\otimes}^C N_2).
\end{aligned}$$

$\square$

Side remark: an alternative proof of this result can be based on the fact that pull-backs commute with equalizers (by the result on page 227 of [ML]), since the intersection of two subspaces can be seen as a pull-back.

We next consider quotient comodules. If N is a left C-subcomodule of a left C-comodule B, then the quotient space B/N inherites a left C-comodule structure as below

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & C\otimes B \arrow[d] & \\
B/N \arrow[r, dashed] & (C\otimes B)/(C\otimes N) \arrow[r, "\cong"] & C\otimes(B/N)
\end{tikzcd} ,$$

and $0 \to N \to B \to B/N \to 0$ is an exact sequence of comodules.

Proposition A.1.1. *Let C be a k-coalgebra, A a right C-comodule and B a left one. Let M be a left C-subcomodule of A and N a right C-subcomodule of B, such that all A, M and B/N are flat as C-comodules. Then there is a canonical isomorphism*

$$(A/M)\otimes^C(B/N) \cong (A\otimes^C B)/(M\otimes^C B + A\otimes^C N) \ .$$

Proof. First, since B/N is flat,

$$(A/M)\otimes^C(B/N) \cong \frac{A\otimes^C(B/N)}{M\otimes^C(B/N)} \ . \tag{*}$$

Now, since A is flat,

$$A\otimes^C(B/N) \cong (A\otimes^C B)/(A\otimes^C N) \ , \tag{**}$$

and since M is flat

$$M\otimes^C(B/N) \cong (M\otimes^C B)/(M\otimes^C N) \ .$$

But according to lemma A.1.1, $M\otimes^C N = (M\otimes^C B) \cap (A\otimes^C N)$. Hence

$$M\otimes^C(B/N) \cong (M\otimes^C B)/\Big((M\otimes^C B) \cap (A\otimes^C N)\Big) \cong (M\otimes^C B + A\otimes^C N)/(A\otimes^C N) \ . \tag{***}$$

From (*), (**) and (***) it follows that

$$(A/M)\otimes^C(B/N) \cong \frac{(A\otimes^C B)/(A\otimes^C N)}{(M\otimes^C B + A\otimes^C N)/(A\otimes^C N)} \cong (A\otimes^C B)/(M\otimes^C B + A\otimes^C N) \ .$$

□

A.2 Assorted lemmas

If C is a k-coalgebra, then its dual $C^* = \mathsf{Hom}_k(C,k)$ becomes a k-algebra with multiplication determined by

$$(f \cdot g)(c) = f(c_1)g(c_2) \ \forall \ f, g \in C^*, \ c \in C, \text{ where } \Delta_C(c) = c_1 \otimes c_2,$$

and with unit element $\epsilon_C \in C^*$.

Lemma A.2.1. *Let C be a k-coalgebra and (M, s) a right C-comodule. Then M becomes a left C^*-module under*

$$f \cdot m = f(m_1)m_0 \text{ where } s(m) = m_0 \otimes m_1 \in M \otimes C, \ \forall \ f \in C^*, m \in M \ .$$

This defines a fully-faithful functor $\mathsf{Comod}^r C \to \mathsf{Mod}^l C^*$, which is an equivalence if C is finite-dimensional.

Proof. See [Mon, lemma 1.6.4] or [Swe, section 2.1]. $\square$

Replacing C by C^{cop} one obtains a version of lemma A.2.1 dealing with the functor $\mathsf{Comod}^l C \to \mathsf{Mod}^r C^*$. There is also a more general version for bicomodules. We will refer to these variants (of this and other results) later in this section or in the main body of this thesis, even though they will not always be explicitly stated.

Remark A.2.1. If R, S and T are k-algebras, U a R-S-bimodule and V a R-T-one, then $\mathsf{Hom}_k(U, V)$ carries a structure of S-T-bimodule, via

$$(s \cdot f)(u) = f(u \cdot s) \text{ and } (f \cdot t)(u) = f(u) \cdot t$$

$$\forall \ f \in \mathsf{Hom}_k(U, V), \ s \in S, \ t \in T \text{ and } u \in U \ .$$

Moreover, $\mathsf{Hom}^l_R(U, V)$ is an S-T-subbimodule of $\mathsf{Hom}_k(U, V)$.

The analogous construction for bicomodules can be performed with the aid of lemma A.2.1, under some finite-dimensionality assumptions:

Lemma A.2.2. *Let C, D and E be k-coalgebras, U a C-D-bicomodule and V a E-D-one. If C and E are finite-dimensional, then $\mathsf{Hom}_k(U,V)$ carries a natural structure of E-C-bicomodule. Moreover, $\mathsf{Hom}^r_D(U,V)$ is a E-C-subbicomodule of $\mathsf{Hom}_k(U,V)$.*

Proof. The result follows trivially from remark A.2.1 and lemma A.2.1. □

Remark A.2.2. Assume the same hypothesis as in remark A.2.1. Then the dual space U^* is an S-R-bimodule via

$$(s\cdot f)(u)=f(u\cdot s) \text{ and } (f\cdot r)(u)=f(r\cdot u)\ \forall\ f\in U^*,\ s\in S,\ r\in R \text{ and } u\in U\ .$$

Moreover, the canonical map

$$d: U^*\otimes V \hookrightarrow \mathsf{Hom}_k(U,V) \text{ defined by } d(f\otimes v)(u)=f(u)v$$

is a morphism of S-T-bimodules.

Lemma A.2.3. *In addition to the hypothesis of lemma A.2.2, assume that D is finite-dimensional. Then U^* is a D-C-bicomodule, the canonical map*

$$d: V\otimes U^* \hookrightarrow \mathsf{Hom}_k(U,V)$$

is a morphism of E-C-bicomodules, and d restricts to a morphism of E-C-bicomodules

$$V\otimes^D U^* \hookrightarrow \mathsf{Hom}^r_D(U,V)\ ,$$

which is an isomorphism if U is finite-dimensional.

Proof. We endow U^* with a structure of D-C-bicomodule by using the previous results as illustrated below:

$$^{C}U^{D} \xrightarrow{\text{lemma A.2.1}} {}_{D^*}U_{C^*} \xrightarrow{\text{remark A.2.2}} {}_{C^*}(U^*)_{D^*} \xrightarrow{\text{lemma A.2.1}} {}^{D}(U^*)^{C} \ .$$

Explicitly, $t : U^* \to D\otimes U^*$, $t(f) = f_{-1}\otimes f_0$ and $s : U \to U\otimes D$, $s(u) = u_0\otimes u_1$ are related by

$$f(u_0)u_1 = f_0(u)f_{-1} \in D \ . \qquad (*)$$

By remark A.2.2 and lemma A.2.1, $d : V\otimes U^* \hookrightarrow \mathsf{Hom}_k(U,V)$ is a morphism of S-T-bicomodules. This map is an isomorphism when U is finite-dimensional. Thus,the proof will be complete when we show that, for any $\alpha \in V\otimes U^*$,

$$\alpha \in V\otimes^{D}U^* \Leftrightarrow d(\alpha) \in \mathsf{Hom}^r_D(U,V) \ .$$

Write $\alpha = v\otimes f$ (we really mean $\sum_i v_i\otimes f_i$, but this abuse of notation is harmless). We have

$$\begin{aligned} \alpha \in V\otimes^{D}U^* &\Leftrightarrow s(v)\otimes f = v\otimes t(f) \\ &\Leftrightarrow s(v)\otimes f = v\otimes f_{-1}\otimes f_0 \Leftrightarrow \forall\ u \in U,\ f(u)s(v) = f_0(u)v\otimes f_{-1} \\ &\overset{(*)}{\Leftrightarrow} \forall\ u \in U,\ f(u)s(v) = f(u_0)v\otimes u_1 \ . \end{aligned}$$

On the other hand,

$$\begin{aligned} &d(\alpha) \in \mathsf{Hom}^r_D(U,V) \Leftrightarrow\ \forall\ u \in U,\ (d(\alpha)\otimes \mathsf{id}_D)s(u) = sd(\alpha)(u) \\ &\Leftrightarrow\ \forall\ u \in U,\ (d(\alpha)\otimes \mathsf{id}_D)(u_0\otimes u_1) = s(f(u)v) \Leftrightarrow\ \forall\ u \in U,\ f(u_0)v\otimes u_1 = f(u)s(v) \ , \end{aligned}$$

which is the same condition. Thus $\alpha \in V\otimes^{D}U^* \Leftrightarrow d(\alpha) \in \mathsf{Hom}^r_D(U,V)$ and the proof is complete. □

Finally, we discuss iterated tensor coproducts and tensor products. Assume that C and D are k-coalgebras and, changing notation slightly, U is a right C-comodule, V a C-D-bicomodule and W a left D-module. In view of lemma A.2.1, V and W are also right and left D^{*op}-modules respectively. By the same lemma, $U \otimes^C V$ is a right D^{*op}-submodule of $U \otimes V$ (since it is a right D-subcomodule). Also, $V \otimes_{D^{*op}} W$ is a quotient left C-comodule of $V \otimes W$ (since it is a quotient left C^{*op}-module). Hence, the following k-spaces are defined

$$(U \otimes^C V) \otimes_{D^{*op}} W \quad \text{and} \quad U \otimes^C (V \otimes_{D^{*op}} W)\ .$$

By general reasons [ML, IX.2] there is always a canonical map

$$(U \otimes^C V) \otimes_{D^{*op}} W \longrightarrow U \otimes^C (V \otimes_{D^{*op}} W)\ .$$

Explicitly, this map exists because

$$\begin{aligned} U \otimes^C (V \otimes_{D^{*op}} W) &= \mathsf{Eq}_k \Big(\ U \otimes (V \otimes_{D^{*op}} W) \underset{\mathsf{id}_U \otimes (t_V \otimes \mathsf{id}_W)}{\overset{s_U \otimes \mathsf{id}}{\rightrightarrows}} U \otimes C \otimes (V \otimes_{D^{*op}} W)\ \Big) \\ &= \mathsf{Eq}_k \Big(\ (U \otimes V) \otimes_{D^{*op}} W \underset{(\mathsf{id}_U \otimes t) \otimes \mathsf{id}_W}{\overset{(s_U \otimes \mathsf{id}_V) \otimes \mathsf{id}_W}{\rightrightarrows}} (U \otimes C \otimes V) \otimes_{D^{*op}} W)\ \Big) \end{aligned}$$

and

$$(U \otimes^C V) \otimes_{D^{*op}} W \xrightarrow{\mathsf{can} \otimes \mathsf{id}_W} (U \otimes V) \otimes_{D^{*op}} W$$

maps into this equalizer by functoriality.

Lemma A.2.4. *In the above situation, if either U is flat as right C-comodule or W is flat as left D^{*op}-module, then the canonical map*

$$(U \otimes^C V) \otimes_{D^{*op}} W \longrightarrow U \otimes^C (V \otimes_{D^{*op}} W)\ .$$

is an isomorphism.

Proof. If U is flat as right C-comodule then $U \otimes^C (-)$ preserves the coequalizer

$$V \otimes_{D^{*op}} W = \mathsf{Coeq}_k(\ V \otimes D^{*op} \otimes W \rightrightarrows V \otimes W\ )$$

so

$$\begin{aligned} U \otimes^C (V \otimes_{D^{*op}} W) &= \mathsf{Coeq}_k(\ U \otimes^C (V \otimes D^{*op} \otimes W) \rightrightarrows U \otimes^C (V \otimes W)\ ) \\ &= \mathsf{Coeq}_k(\ (U \otimes^C V) \otimes D^{*op} \otimes W) \rightrightarrows (U \otimes^C V) \otimes W)\ ) \\ &= (U \otimes^C V) \otimes_{D^{*op}} W\ . \end{aligned}$$

The proof is similar if W is flat as left D^{*op}-module. ☐

Notice also that if U is a C'-C-bicomodule and W a D-D'-bicomodule, then the canonical map above is a morphism of C'-D'-bicomodules.

A.3 Products and coproducts of comodules

Arbitrary products in the category of left comodules over a k-coalgebra C exist. They are described in terms of products of the corresponding modules over C^* in the final note to [T2]. A more direct description is as follows.

Proposition A.3.1. *Let C be a k-coalgebra and $\{(M_i, t_i)\}_{i \in I}$ an arbitrary family of left C-comodules. Then the product $\prod^C_{i \in I} M_i$ in the category $\mathsf{Comod}^l C$ exists and is*

$$\prod\nolimits^C_{i \in I} M_i = \Big\{ m = \{m_i\} \in \prod_{i \in I} M_i \ \Big/ \begin{array}{c} \text{there is a finite dimensional subcoalgebra} \\ C_m \text{ of } C \text{ such that } t_i(m_i) \in C_m \otimes M_i \ \forall\ i \in I \end{array} \Big\},$$

where $\prod_{i \in I} M_i$ is the product in the category $\mathtt{Vec}_k$.

Proof. First we show that $M = \prod_{i\in I}^{C} M_i$ is in fact a left C-comodule, as follows. Given $m = \{m_i\} \in M$, let $\{c_j\}_{j\in J}$ be a (finite) k-basis of C_m, and write, for each $i \in I$,

$$t_i(m_i) = \sum_{j\in J} c_j \otimes m_{ij} \text{ with } m_{ij} \in M_i \ \forall\ j \in J.$$

Coassociativity for each t_i implies that, for each $j \in J$, the element $\{m_{ij}\}_{i\in I}$ belongs to M. Thus we can define $t : M \to C \otimes M$ by setting

$$t(m) = \sum_{j\in J} c_j \otimes \{m_{ij}\}_{i\in I}.$$

This definition is clearly independent of the subcoalgebra C_m chosen. Coassociativity and counitality for t follow from those for t_i.

Let $p_i : M \to M_i$ denote the restriction of the canonical projection $\prod_{i\in I} M_i \to M_i$. By construction, p_i is a morphism of left C-comodules. Let us check the universal property of the product for (M, p_i). Let (N, t) be a left C-comodule and $f_i : N \to M_i$ morphisms of left C-comodules for each $i \in I$. Let $f : N \to \prod_{i\in I} M_i$, $f(n) = \{f_i(n)\}$, be the corresponding map into the product in Vec_k. We claim that $f(n) \in M$. To see this, write $t(n) = \sum_k c_k \otimes n_k$ for some (finitely-many) $c_k \in C$. Let C_n be the finite-dimensional subcoalgebra of C spanned by the c_k's (this is possible by the finiteness theorem [Mon, 5.1.1.2]). Then, for each $i \in I$,

$$t_i(f_i(n)) = (\mathrm{id}_C \otimes f_i)t(n) = \sum_k c_k \otimes f_i(n_k) \in C_n \otimes M_i,$$

which shows that $f(n) \in M$. This thus defines a linear map $f : N \to M$ such that $p_i f = f_i \ \forall\ i \in I$. This latter property clearly determines f uniquely. Moreover, by definition of $t : M \to C \otimes M$, we have

$$t(f(n)) = \sum_k c_k \otimes \{f_i(n_k)\} = \sum_k c_k \otimes f(n_k) = (\mathsf{id}_C \otimes f)t(n),$$

i.e. f is a morphism of left C-comodules. This completes the proof. $\square$

The following consequence of the above result was quoted in section 6.3.

Corollary A.3.1. *Let C be a k-coalgebra. Then the forgetful functor $\mathsf{Comod}^l C \to \mathit{Vec}_k$ has a left adjoint if and only if C is finite-dimensional.*

Proof. If C is finite-dimensional, then there is an equivalence $\mathsf{Comod}^l C \cong \mathsf{Mod}^r C^*$ (lemma A.2.1), which preserves the forgetful functors to Vec_k. But for any k-algebra A, the forgetful functor $\mathsf{Mod}^r A \to \mathit{Vec}_k$ has a left adjoint, namely the functor $(-) \otimes A : \mathit{Vec}_k \to \mathsf{Mod}^r A$.

Conversely, assume that $\mathsf{Comod}^l C \to \mathit{Vec}_k$ has a left adjoint. Then, by theorem V.5.1 in [ML], this functor preserves all products. Let $\{c_i\}_{i \in I}$ be a k-basis of C. We will show that it is finite by considering products indexed by I. In fact, for each $i \in I$ let $M_i = C$ with its usual left C-comodule structure, and consider $M = \prod_{i \in I}^C M_i$. By assumption, there is a linear isomorphism $M \cong \prod_{i \in I} M_i$ preserving the canonical projections. It follows that this isomorphism must be the canonical inclusion $M \hookrightarrow \prod_{i \in I} M_i$. In particular, $\{c_i\} \in M$. This means that there is a finite-dimensional subcoalgebra C_0 of C such that

$$\Delta_C(c_i) \in C_0 \otimes C \ \forall \ i \in I \ .$$

But then

$$c_i = (\mathsf{id}_C \otimes \epsilon_C)\Delta_C(c_i) \in C_0 \ \forall \ i \in I,$$

from where $C = C_0$, i.e. C is finite-dimensional. $\square$

Similarly, the forgetful functor $\{X\text{-graded sets}\} \to \mathit{Sets}$ has a left adjoint if and only if $X = \{*\}$, since the product of two X-graded sets M and N is simply their tensor coproduct $M \times^X N$, which is not preserved by the forgetful functor unless $X = \{*\}$.

Notice that by proposition 3.0.1, the forgetful functor $\mathsf{Comod}^l C \to \mathit{Vec}_k$ always has a right adjoint, namely the coinduction functor $C \otimes (-) : \mathit{Vec}_k \to \mathsf{Comod}^l C$.

Next, we discuss coproducts of comodules. Let $\{(M_i, t_i)\}_{i \in I}$ be an arbitrary family of left comodules over a k-coalgebra C. Let $M = \oplus_{i \in I} M_i$. Given $m = \{m_i\} \in M$, write for each $i \in I$,

$$t_i(m_i) = \sum_{j \in J} c_j \otimes m_{ij} \text{ with } m_{ij} \in M_i \ \forall\ j \in J.$$

For each $j \in J$, the family $\{m_{ij}\}_{i \in I}$ is finite, since so is the family $\{m_i\}_{i \in I}$. Thus we can define $t : M \to C \otimes M$ by setting

$$t(m) = \sum_{j \in J} c_j \otimes \{m_{ij}\}_{i \in I}.$$

It is clear that (M, t) is then a left C-comodule, and that together with the canonical inclusions $M_i \hookrightarrow M$ it becomes the coproduct of the family $\{M_i\}$ in $\mathsf{Comod}^l C$. There are linear inclusions

$$\bigoplus_{i \in I} M_i \hookrightarrow \prod\nolimits_{i \in I}^{C} M_i \hookrightarrow \prod_{i \in I} M_i \ .$$

The coproduct M is also called the direct sum of the comodules $\{M_i\}$.

Finally, we discuss direct sums in relation to flatness.

Lemma A.3.1. *Let M be the direct sum of a family $\{M_i\}$ of left C-comodules. Then M is flat if and only if M_i is flat $\forall\ i \in I$.*

Proof. If N is a right C-comodule, then

$$N{\otimes^C}\Big(\bigoplus_{i\in I} M_i\Big) \cong \bigoplus_{i\in I}(N{\otimes^C}M_i)\ .$$

Thus, there is an isomorphism of functors $(-){\otimes^C}M \cong \bigoplus_{i\in I}\Big((-){\otimes^C}M_i\Big)$, from where the result follows. $\square$

Lemma A.3.2. *Let*

$$0 \to J \to A \to A/J \to 0$$

be an exact sequence of left C-comodules, where J is flat. Then A is flat if and only if A/J is flat.

Proof. Since J is flat=injective, the sequence splits. Thus, $A \cong J \oplus A/J$ as left C-comodules. The result now follows from lemma A.3.1. $\square$

Lemma A.3.3. *Let M_1 and M_2 be subcomodules of a left C-comodule M. Suppose that M_1, M_2 and $M_1 \cap M_2$ are flat. Then so is $M_1 + M_2$.*

Proof. By lemma A.3.1, $M_1 \oplus M_2$ is flat. The result now follows from lemma A.3.2, applied to the exact sequence

$$0 \to M_1 \cap M_2 \to M_1 \oplus M_2 \to M_1 + M_2 \to 0\ .$$

$\square$

Lemma A.3.4. *A left comodule is flat if and only if it is a direct summand of a free left comodule.*

Proof. Let (M, t) be a flat left C-comodule. Then $t : M \to C \otimes M$ is a morphism of left C-comodules, where we view $C \otimes M$ as left C-comodule via $\Delta_C \otimes \mathsf{id}_M$ (thus, $C \otimes M$ is free by definition). The map t is injective since it is split by the k-linear map $\epsilon_C \otimes \mathsf{id}_M : C \otimes M \to M$. Since M is flat=injective, t also admits a splitting of left C-comodules. Therefore M is a direct summand of $C \otimes M$.

The converse implication follows from lemma A.3.1, since free comodules are flat, as explained in section A.1. □

For the following result, recall the definition of tensor product of comodules from section 7.1.

Lemma A.3.5. *Let A be a flat left C-comodule and B a flat left D-comodule. Then $A \otimes B$ is flat as left $C \otimes D$-comodule.*

Proof. By lemma A.3.4, A is a direct summand of some $C \otimes V$, and B a direct summand of some $D \otimes W$. It follows that $A \otimes B$ is a direct summand of $C \otimes D \otimes V \otimes W$, hence flat. □

Appendix B

Braids and q-binomials

B.1 Introduction

The classical identities between the q-binomial coefficients and factorials can be generalized to a context where numbers are replaced by braids, or more precisely, elements of the braid group algebras kB_n. Thus, for every pair i, n of natural numbers there is defined an element $b_i^{(n)} \in kB_n$ (section B.3), and these satisfy analogs of the classical identities for the binomial coefficients (sections B.4 through B.8). Moreover, by choosing representations of the braid groups one obtains concrete realizations of these identities; the simplest such choices yield the identities for the classical and q-binomial coefficients, other choices yield new identities that involve matrices rather than numbers.

Table B.1 describes the action of the braids introduced in this appendix when X is certain one-dimensional representation defined by $q \in k^*$ (section B.2.5). The definition of the q-analogs will be reviewed before each corresponding braid analog is introduced.

These *binomial* braids $b_i^{(n)}$ play a crucial role in the generalization of the definition of the quantum group $U_q^+(C)$ of Drinfeld [Dr1] and Jimbo [Jim] presented in section 9.8 of the main body of this thesis. In this appendix we concentrate on their combinatorial properties.

At the level of braids, the proofs of the combinatorial identities follow a constant pattern: first there is the set-theoretic part, which involves dealing with the same bijections that are used for the case of the classical ($q = 1$) identities, then there is the geometric part that consists in proving that two braids, labeled by corresponding elements under the bijection considered, are in fact equal.

The classical q-identities that we generalize are taken mostly from papers by Goldman and Rota [GR]; in particular these include Pascal's, Vandermonde's and Cauchy's identities, the factorial formula, Rota's binomial theorem, Möbius inversion, several identities involving multinomial braids and definitions and formulas for the Galois, Fibonacci and Catalan braids.

It is also possible to define the braid analog of a partition of a set, and then Stirling and Bell braids. These will be studied elsewhere.

This appendix reproduces [A].

Table B.1: Combinatorial braids and q-analogs

Braid	name	defined in section	action
$s_i^{(n)}$	generator	B.2.1	q
$s^{(n)}(i,j)$		B.2.1	q^{j-i}
$c^{(n)}$	twistor	B.2.2	$q^{\binom{n}{2}}$
$\beta_{m,n}$	braiding	B.2.4	q^{mn}
$s_I^{(n)}$		B.3	$q^{\|I\|}$
$b_i^{(n)}$	binomial	B.3	$\left[{n \atop i} \right]$
$[n]$	natural	B.5	$[n]$
$s_\sigma^{(n)}$		B.5	$q^{\mathsf{inv}(\sigma)}$
$f^{(n)}$	factorial	B.5	$[n]!$
$s_f^{(n)}$		B.7.1	$q^{\mathsf{inv}(f)}$
$m^{(\eta)}$	multinomial	B.7.1	$\left[{n \atop \eta} \right]$
$\mu^{(n)}$	Möbius	B.6.2	$(-1)^n q^{\binom{n}{2}}$
$C^{(n)}$	Catalan	B.8	C_n
$G^{(n)}$	Galois	B.8	G_n
$F^{(n)}$	Fibonacci	B.8	F_n

B.2 Braid groups and the braid category

B.2.1 Basics

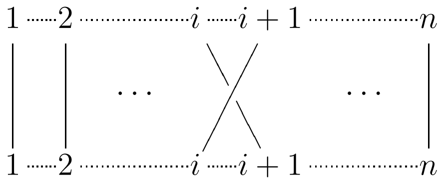
The group B_n of *braids in n strands* has generators $s_1^{(n)}, \dots, s_{n-1}^{(n)}$ subject to the relations

$$s_i^{(n)} s_j^{(n)} = s_j^{(n)} s_i^{(n)} \qquad \text{if } |i-j| \geq 2, \tag{A1}$$

$$s_i^{(n)} s_{i+1}^{(n)} s_i^{(n)} = s_{i+1}^{(n)} s_i^{(n)} s_{i+1}^{(n)} \qquad \text{if } 1 \leq i \leq n-2. \tag{A2}$$

The generator $s_i^{(n)}$ is represented by the following picture, and the product st of two braids s and t in B_n is obtained by putting the picture of s on top of that of t. The identity of B_n is represented by the picture with n vertical strands; the inverse of s is obtained by reflecting its picture across a horizontal line, without leaving the plane of the picture.

$s_i^{(n)} =$



The collection $\mathfrak{B} = \coprod_{n\geq 0} B_n$ of all braid groups forms a category, where the objects are the natural numbers, B_n is the set of endomorphisms of n, and there are no morphisms between distinct objects. This category is monoidal; the tensor product $s \otimes t \in B_{n+m}$ of two braids $s \in B_n$ and $t \in B_m$ is obtained by putting t to the right of s, i.e. $s_i^{(n)} \otimes s_j^{(m)} = s_i^{(n+m)} s_{n+j}^{(n+m)}$. Moreover, this monoidal category is *braided*, in the sense that there is a natural map $\beta_{n,m} : n{\otimes}m \to m{\otimes}n$, i.e. a braid $\beta_{n,m} \in B_{n+m}$, satisfying some axioms (B.2.4 below). For more details on this, the reader is referred to [K], X.6 and XIII.2.

We develop some basic notation. For each pair (i,j) with $1 \leq i \leq j \leq n$, define

$$s^{(n)}(i,j) = \begin{cases} 1 & \text{if } i = j, \\ s_i^{(n)} s_{i+1}^{(n)} \cdots s_{j-1}^{(n)} & \text{if } i < j. \end{cases}$$

We provide a first set of lemmas.

Lemma.

$$s^{(n)}(i,k) = s^{(n)}(i,j)s^{(n)}(j,k) \qquad \text{when } 1 \leq i \leq j \leq k \leq n \qquad (1)$$

$$s_i^{(m+n)} = s_i^{(m)}{\otimes}1^{(n)} \qquad \text{when } 1 \leq i \leq m-1,\ n \geq 0 \quad (2)$$

$$s^{(m+n)}(i,j) = s^{(m)}(i,j){\otimes}1^{(n)} \qquad \text{when } 1 \leq i \leq j \leq m,\ n \geq 0$$

$$s_{i+n}^{(m+n)} = 1^{(n)}{\otimes}s_i^{(m)} \qquad \text{when } 1 \leq i \leq m-1,\ n \geq 0 \quad (3)$$

$$s^{(m+n)}(i+n, j+n) = 1^{(n)}{\otimes}s^{(m)}(i,j) \qquad \text{when } 1 \leq i \leq j \leq m,\ n \geq 0$$

$$s_{i+l}^{(m+n)} = 1^{(l)}{\otimes}s_i^{(m)}{\otimes}1^{(n-l)} \qquad \text{when } 1 \leq i \leq m-1,\ 0 \leq l \leq n \qquad (4)$$

$$s^{(m+n)}(i+l, j+l) = 1^{(l)}{\otimes}s^{(m)}(i,j){\otimes}1^{(n-l)} \qquad \text{when } 1 \leq i \leq j \leq m,\ 0 \leq l \leq n$$

$$s^{(n)}(i,j)s_h^{(n)} = s_{h+1}^{(n)}s^{(n)}(i,j) \qquad \text{when } 1 \leq i \leq h \leq j-2 \qquad (5)$$

$$s^{(n)}(i,j)s^{(n)}(h,k) = s^{(n)}(h+1,k+1)s^{(n)}(i,j) \qquad \text{when } 1 \leq i \leq h \leq k \leq j-1\ .$$

Proof. Equation (1) is a direct consequence of the notation, the first parts of (2) and (3) hold simply by definition of the tensor product, and the second parts follow by repeated use of the first ones. Now,

$$s_{i+l}^{(m+n)} \overset{(2)}{=} s_{i+l}^{(m+l)} \otimes 1^{(n-l)} \overset{(3)}{=} 1^{(l)} \otimes s_i^{(m)} \otimes 1^{(n-l)} \ ,$$

proving the first part of (4). Similarly the second part follows from the second parts of (2) and (3).

Finally, if $1 \le i \le h \le j-2$, we can write

$$s^{(n)}(i,j)s_h^{(n)} \overset{(1)}{=} s^{(n)}(i,h)s_h^{(n)}s_{h+1}^{(n)}s^{(n)}(h+2,j)s_h^{(n)} \overset{(A1)}{=}$$

$$s^{(n)}(i,h)s_h^{(n)}s_{h+1}^{(n)}s_h^{(n)}s^{(n)}(h+2,j) \overset{(A2)}{=}$$

$$s^{(n)}(i,h)s_{h+1}^{(n)}s_h^{(n)}s_{h+1}^{(n)}s^{(n)}(h+2,j) \overset{(A1)}{=}$$

$$s_{h+1}^{(n)}s^{(n)}(i,h)s_h^{(n)}s_{h+1}^{(n)}s^{(n)}(h+2,j) \overset{(1)}{=} s_{h+1}^{(n)}s^{(n)}(i,j) \ ,$$

which proves the first part of (5); now for the second notice that if $k = h$ then there is nothing to prove; otherwise $j > k > h$ so it follows by repeated use of the first.

□

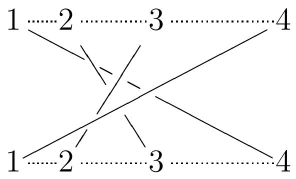
B.2.2 Vertical symmetry

There is an involution $\widetilde{\ }: B_n \to B_n$ defined by $\widetilde{s_i^{(n)}} = s_{n-i}^{(n)}$. The picture for $\tilde{s}$ is obtained by rotating in 3-space that of s 180 degrees around a vertical line. Consider the *twistor* braid,

$$c^{(n)} = s^{(n)}(1,n)s^{(n)}(1,n-1)\cdots s^{(n)}(1,2)s^{(n)}(1,1) \ .$$

For instance

$$c^{(4)} \;=$$



Repeated use of (A1) and (5) shows that $c^{(n)}s_i^{(n)} = s_{n-i}^{(n)}c^{(n)}$, hence $\widetilde{\ }$ is the inner automorphism defined by conjugation by $c^{(n)}$. It follows that ${c^{(n)}}^2$ is in the center of the braid group, since $\tilde{\tilde{s}} = s$ for any s. Moreover, it can be shown that ${c^{(n)}}^2$ generates $Z(B_n)$; we won't make use of this fact.

Let us prove that, for any $s \in B_n$ and $t \in B_m$,

$$\widetilde{s\otimes t} = \tilde{t}\otimes\tilde{s}\ . \tag{6}$$

Proof. Notice that if the statement holds for $s\otimes t$ and $s'\otimes t'$, then so it does for $ss'\otimes tt'$. Hence it suffices to prove it for $s = s_i^{(n)}$ and $t = s_j^{(m)}$. Now,

$$s_i^{(n)}\otimes s_j^{(m)} \overset{(2),\,(3)}{=} s_i^{(n+m)}\, s_{n+j}^{(n+m)} \Rightarrow$$

$$\widetilde{s_i^{(n)}\otimes s_j^{(m)}} = s_{n+m-i}^{(n+m)} s_{n+m-(n+j)}^{(n+m)} = s_{n+m-i}^{(n+m)} s_{m-j}^{(n+m)} \overset{(A1)}{=} \text{ (since } n+m-i \geq m+1)$$

$$s_{m-j}^{(n+m)}\, s_{n+m-i}^{(n+m)} \overset{(2),\,(3)}{=} s_{m-j}^{(m)}\otimes s_{n-i}^{(n)} = \widetilde{s_j^{(m)}}\otimes\widetilde{s_i^{(n)}}\ .$$

□

B.2.3 Horizontal symmetry

There is a map $* : B_n \to B_n$ defined by the conditions that ${s_i^{(n)}}^* = s_i^{(n)}$ and $(st)^* = t^*s^*$. The picture for s^* is obtained by rotating that of s in 3-space 180 degrees around a horizontal line.

It is clear that the three operators $^*, \tilde{}, ^{-1} : B_n \to B_n$ commute pairwise, and also that

$$
\begin{aligned}
& s^{**} = s && \forall\ s \in B_n, \\
& (s{\otimes}t)^* = s^*{\otimes}t^* && \forall\ s \in B_n,\ t \in B_m, && (7) \\
& \widetilde{s^{(n)}(i,j)}^* = s^{(n)}(n+1-j, n+1-i) && \forall\ i,j,n, && (8) \\
& {c^{(n)}}^* = c^{(n)} && \forall\ n. && (9)
\end{aligned}
$$

From (9) it follows easily that

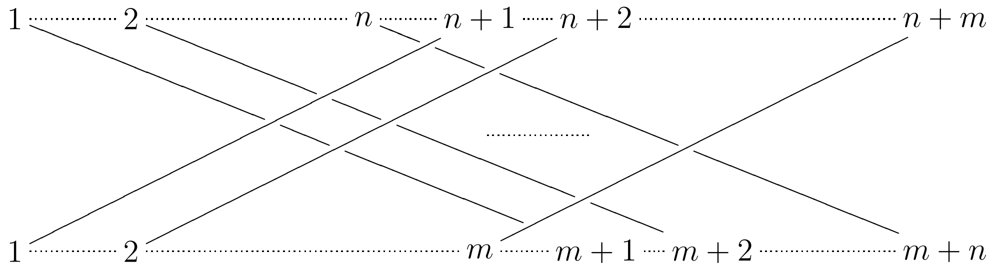
$$c^{(n)} = s^{(n)}(n,n)s^{(n)}(n-1,n)\dots s^{(n)}(2,n)s^{(n)}(1,n)$$

and from here that

$${c^{(n)}}^2 = s^{(n)}(1,n)^n.$$

B.2.4 Properties of the braiding

The *braiding* $\beta_{m,n}$ is most easily defined in terms of its picture:



It is viewed as a natural map $\beta_{m,n} : m{\otimes}n \to n{\otimes}m$ in the category $\mathfrak{B}$ of braids, and as such it satisfies some important properties. We will list some of them below without proof, since we won't use them, although they are very easily obtained through the use of pictures, see [K] XIII.2. However, it will be convenient for us to have an explicit description of $\beta_{m,n}$ in terms of the canonical generators. For this, we first define some special "powers" for braids as follows.

Let $m \geq 1$. For $s \in B_m$ and $n \geq 0$, define

$$s^{\langle n\rangle} = \begin{cases} 1 & \text{if } n = 0, \\ s & \text{if } n = 1, \\ 1^{(n-1)}{\otimes}s \cdot 1^{(n-2)}{\otimes}s{\otimes}1 \cdot \ldots \cdot 1{\otimes}s{\otimes}1^{(n-2)} \cdot s{\otimes}1^{(n-1)} & \text{if } n \geq 2. \end{cases} \tag{10}$$

Thus $s^{\langle n\rangle} \in B_{m+n-1}\ \forall\ m \geq 1,\ n \geq 0$ (and it is not defined if $m = 0$). Notice that $s^{\langle n+1\rangle} = 1{\otimes}s^{\langle n\rangle} \cdot s{\otimes}1$, from here it follows easily by induction that

$$\begin{aligned} & s^{\langle p+q\rangle} = 1^{(q)}{\otimes}s^{\langle p\rangle} \cdot s^{\langle q\rangle}{\otimes}1^{(p)} && \forall\ p, q \geq 0, \\ & 1^{(k)}{\otimes}s^{\langle n\rangle}{\otimes}1^{(h)} = \left[1^{(k)}{\otimes}s{\otimes}1^{(h)}\right]^{\langle n\rangle} && \forall\ n, k, h \geq 0\ . \end{aligned} \tag{11}$$

We then define

$$\beta_{m,n} = s^{(m+1)}(1, m+1)^{\langle n\rangle}\ \in\ B_{m+n}\ . \tag{12}$$

It is easy to see that this corresponds to the picture above. These are some of the properties that β satisfies:

$$\begin{aligned}
&\beta_{m,n} \cdot s{\otimes}t = t{\otimes}s \cdot \beta_{m,n} && \forall\ s \in B_m,\ t \in B_n,\ (\textit{naturality} \text{ of the braiding}),\\
&c^{(n+m)} = c^{(n)}{\otimes}c^{(m)} \cdot \beta_{m,n} && \forall\ m,n \geq 0,\\
&\widetilde{\beta_{m,n}} = \beta_{n,m} = \beta^*_{m,n} && \forall\ m,n \geq 0,\\
&\beta_{p,q+r} = 1^{(q)}{\otimes}\beta_{p,r} \cdot \beta_{p,q}{\otimes}1^{(r)} && \forall\ p,q,r \geq 0,\\
&\beta_{p+q,r} = \beta_{p+q,r}{\otimes}1^{(q)} \cdot 1^{(p)}{\otimes}\beta_{q,r} && \forall\ p,q,r \geq 0.
\end{aligned}$$

B.2.5 Representations

Throughout the appendix k will denote a fixed field (although any commutative ring would do just as well).

The identities we will obtain between elements of the braid group algebras kB_n can be converted into matrix or numerical identities by choosing k-linear representations of the braid groups B_n.

More precisely we will be interested in *monoidal representations* of the braid category $\mathfrak{B}$, that is a vector space X, such that the braid group B_n acts on the tensor power $X^{\otimes n}$, with the property that

$$s{\otimes}t \cdot x{\otimes}y = (s \cdot x){\otimes}(t \cdot y)\ \forall\ s \in B_n,\ t \in B_m,\ x \in X^{\otimes n},\ y \in X^{\otimes m}\ .$$

Since $s_i^{(n)} = 1^{(i-1)}{\otimes}s_1^{(2)}{\otimes}1^{(n-i+1)}$, this condition implies that the action of B_n on $X^{\otimes n}$ is uniquely determined by the action of $s_1^{(2)}$ on $X{\otimes}X$. Moreover, a linear operator $R : X{\otimes}X \to X{\otimes}X$ defines a monoidal representation of $\mathfrak{B}$ if and only if it is invertible and satisfies the *Yang-Baxter equation*:

$$(R{\otimes}\mathsf{id}_X) \circ (\mathsf{id}_X{\otimes}R) \circ (R{\otimes}\mathsf{id}_X) = (\mathsf{id}_X{\otimes}R) \circ (R{\otimes}\mathsf{id}_X) \circ (\mathsf{id}_X{\otimes}R) \ .$$

This is a consequence of (A2).

If X is one-dimensional, then any invertible operator $R : X \to X$ satisfies this equation. R is necessarily given by multiplication by some non-zero scalar $q \in k$. Hence, in this case, $s_i^{(n)}$ acts by multiplication by q for every $n \geq 2,\ 1 \leq i \leq n-1$. It is this simplest choice that will produce the classical q-identities from the identities for braids. In particular the trivial one-dimensional representation yields the case $q = 1$. Higher dimensional representations are are discussed in sections 9.8 and B.9. In this regard we should add that Majid began the study of combinatorial identities between operators on tensor powers of a vector space X corresponding to a Yang-Baxter operator on $X{\otimes}X$: in thm. 10.4.12 of [Maj] the case $i = 1$ of (21) is obtained.

The chart in section B.1 describes the action of the braids introduced in this appendix when X is the one-dimensional representation defined by $q \in k^*$ as above.

Let us also remark that since the non-commutativity of the braid groups necessarily disappears when acting on a one-dimensional representation, the actions of s, $\widetilde{s}$ and s^* coincide for any braid s in this case.

B.3 Binomial braids

For each pair (n, i) with $i \leq n$ let $\mathcal{S}_i(n)$ denote the set of subsets of $\{1, 2, \ldots, n\}$ with cardinality i.

Recall that the q-binomial coefficients can be defined as

$$\left[\begin{smallmatrix} n \\ i \end{smallmatrix} \right] = \sum_{I \in \mathcal{S}_i(n)} q^{\|I\|} \text{ where } \|I\| = \sum_{j \in I} j - \sum_{j=1}^{i} j \ .$$

The braid analog of this definition is as follows.

First, for each $I \in \mathcal{S}_i(n)$, write $I = \{j_1, j_2, \ldots, j_i\}$ with $j_1 < j_2 < \ldots < j_i$, then define $s_I^{(n)} \in B_n$ as

$$s_I^{(n)} = s^{(n)}(i, j_i) \cdots s^{(n)}(2, j_2) s^{(n)}(1, j_1) \ ;$$

if $i = 0$ we let $s_\emptyset^{(n)} = 1$.

For instance if $I = \{m+1, m+2, \ldots, m+n\} \in \mathcal{S}_n(m+n)$ then $s_I^{(m+n)} = \beta_{m,n}$.

Then the binomial braid $b_i^{(n)} \in kB_n$ is defined as

$$b_i^{(n)} = \sum_{I \in \mathcal{S}_i(n)} s_I^{(n)} \ .$$

Thus $b_0^{(n)} = b_n^{(n)} = 1 \ \forall \ n$, while for instance

$$b_1^{(2)} = 1 + s_1^{(2)}, \ b_1^{(3)} = 1 + s_1^{(3)} + s_1^{(3)} s_2^{(3)}, \ b_2^{(3)} = 1 + s_2^{(3)} + s_2^{(3)} s_1^{(3)} \ .$$

We see that $b_i^{(n)} \neq b_{n-i}^{(n)}$ in general. However:

Proposition. For all $n \geq i \geq 0$,

$$\widetilde{b_i^{(n)}} = b_{n-i}^{(n)}. \tag{13}$$

Proof. Consider the bijection $\mathcal{S}_i(n) \to \mathcal{S}_{n-i}(n)$ that sends I to $\tilde{I}^c$, where $\tilde{I} = \{n + 1 - i \ / \ i \in I\}$. It is enough to show that, for every $I \in \mathcal{S}_i(n)$,

$$\widetilde{s_I^{(n)}} = s_{\tilde{I}^c}^{(n)} \ . \tag{*}$$

First, we show that if (*) holds when $n \in I$, then it holds for every I. In fact, given $I \in \mathcal{S}_i(n)$, let $m = \max I$, and let I' be the same set I but viewed as en element of $\mathcal{S}_i(m)$. Then we have that

$$s_I^{(n)} \overset{(2)}{=} s_{I'}^{(m)} \otimes 1^{(n-m)},$$

hence, by (6), and assuming (*) for I',

$$\widetilde{s_I^{(n)}} = 1^{(n-m)} \otimes \widetilde{s_{I'}^{(m)}} \overset{(*)}{=} 1^{(n-m)} \otimes s_{\tilde{I'}^c}^{(m)} = 1^{(n-m)} \otimes s_{m+1-I'^c}^{(m)} \overset{(3)}{=} s_{n+1-I^c}^{(n)} = s_{\tilde{I}^c}^{(n)},$$

so (*) holds for I as well.

To finish the proof we show (*) by induction on i. For $i = 0$ it is clear. Assume $i \geq 1$. As just explained, we can also assume that $n \in I$. Therefore, we can decompose $I = I_1 \cup \{n\}$ with $I_1 \in \mathcal{S}_{i-1}(n-1)$; then we have $\tilde{I} = \widetilde{I_1} \cup \{1\}$ and $\widetilde{I_1}^c = \tilde{I}^c \cup \{1\}$.

Write $\tilde{I}^c = \{h_1 < h_2 < \ldots < h_{n-i}\}$, so that $\widetilde{I_1}^c = \{1 < h_1 < h_2 < \ldots < h_{n-i}\}$. We have

$$s_I^{(n)} = s^{(n)}(i,n) s_{I_1}^{(n)} = s_i^{(n)} s_{i+1}^{(n)} \ldots s_{n-1}^{(n)} s_{I_1}^{(n)},$$

hence, by induction hypothesis,

$$\begin{aligned} \widetilde{s_I^{(n)}} &= s_{n-i}^{(n)} s_{n-i-1}^{(n)} \ldots s_1^{(n)} s_{\widetilde{I_1}^c}^{(n)} \\ &= s_{n-i}^{(n)} s_{n-i-1}^{(n)} \ldots s_1^{(n)} s^{(n)}(n-i+1, h_{n-i}) \ldots s^{(n)}(3, h_2) s^{(n)}(2, h_1) s^{(n)}(1,1) \ . \end{aligned}$$

Now using (A1), $s^{(n)}(n-i+1, h_{n-i})$, can be moved to the left past all the factors $s_1^{(n)}$, ... ,$s_{n-i-1}^{(n)}$. Then, it combines with $s_{n-i}^{(n)}$ to form $s^{(n)}(n-i, h_{n-i})$. Similarly the other factors of the form $s^{(n)}(k+1,h_k)$ can be moved to the left until they reach $s_k^{(n)}$ to form $s^{(n)}(k,h_k)$. At the end of the process we have

$$\widetilde{s_I^{(n)}} = s^{(n)}(n-i,h_{n-i})\dots s^{(n)}(2,h_2)s^{(n)}(1,h_1) = s_{\tilde{I}^c}^{(n)} \ .$$

This finishes the induction and the proof. $\square$

B.4 Identities of Pascal and Vandermonde

For the q-binomial coefficients Pascal's identity says that

$$\genfrac{[}{]}{0pt}{}{n}{i} = q^{n-i}\genfrac{[}{]}{0pt}{}{n-1}{i-1} + \genfrac{[}{]}{0pt}{}{n-1}{i} = \genfrac{[}{]}{0pt}{}{n-1}{i-1} + q^i\genfrac{[}{]}{0pt}{}{n-1}{i} \ .$$

Its generalization to braids is as follows.

Proposition. For any $i=1,\dots,n-1$,

$$b_i^{(n)} = s^{(n)}(i,n)\cdot b_{i-1}^{(n-1)}\otimes 1 + b_i^{(n-1)}\otimes 1 = 1\otimes b_{i-1}^{(n-1)} + s^{(n)}\widetilde{(n-i},n)\cdot 1\otimes b_i^{(n-1)} \ . \tag{14}$$

Proof. Consider the bijection $\mathcal{S}_{i-1}(n-1)\cup\mathcal{S}_i(n-1)\to\mathcal{S}_i(n)$ that sends $I\in\mathcal{S}_{i-1}(n-1)$ to $I\cup\{n\}\in\mathcal{S}_i(n)$ and $J\in\mathcal{S}_i(n-1)$ to $J\in\mathcal{S}_i(n)$. From (2) and the definition of s_I we see that

$$s_J^{(n)} = s_J^{(n-1)}\otimes 1 \text{ and } s_{I\cup\{n\}}^{(n)} = s^{(n)}(i,n)\cdot s_I^{(n-1)}\otimes 1 \ ;$$

summing over all such I and J we obtain the first equality. The other one follows by applying $\widetilde{\ }$, using (6) and replacing $n-i$ by i.

$\square$

Vandermonde's identity says that

$$\left[\begin{smallmatrix} m+n \\ p \end{smallmatrix}\right] = \sum_{k=0}^{p} q^{(m-k)(p-k)} \left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right] \left[\begin{smallmatrix} n \\ p-k \end{smallmatrix}\right] .$$

Its generalization to braids reads:

Proposition. For any m, n, p with $0 \leq p \leq m, n$,

$$b_p^{(m+n)} = \sum_{k=0}^{p} 1^{(k)} \otimes \beta_{m-k,p-k} \otimes 1^{(n-p+k)} \cdot b_k^{(m)} \otimes b_{p-k}^{(n)} . \tag{15}$$

Proof. Consider the bijection

$$\bigcup_{k=0}^{p} \mathcal{S}_k(m) \times \mathcal{S}_{p-k}(n) \to \mathcal{S}_p(m+n) \ , \ (I, J) \mapsto I \cup (m+J) \ .$$

It suffices to show that, for each $I \in \mathcal{S}_k(m)$ and $J \in \mathcal{S}_{p-k}(n)$,

$$s_{I \cup (m+J)}^{(m+n)} = 1^{(k)} \otimes \beta_{m-k,p-k} \otimes 1^{(n-p+k)} \cdot s_I^{(m)} \otimes s_J^{(n)} . \tag{*}$$

Let $h = p - k$. If $h = 0$ then (*) reduces to $s_I^{(m+n)} = s_I^{(m)} \otimes 1^{(n)}$, which holds by (2).

Assume $h \geq 1$. Write $I = \{i_1 < \ldots < i_k\}$ and $J = \{j_1 < \ldots < j_h\}$ so that $I \cup (m+J) = \{i_1 < \ldots < i_k < m + j_1 < \ldots < m + j_h\}$. Then

$$\begin{aligned}
s_{I \cup (m+J)}^{(m+n)} &= s^{(m+n)}(k+h, m+j_h) s^{(m+n)}(k+h-1, m+j_{h-1}) \ldots s^{(m+n)}(k+1, m+j_1) \cdot \\
&\quad \cdot s^{(m+n)}(k, i_k) s^{(m+n)}(k-1, i_{k-1}) \ldots s^{(m+n)}(1, i_1) \\
&\overset{(1),(2)}{=} \Big[s^{(m+n)}(k+h, m+h) s^{(m+n)}(m+h, m+j_h) \Big] \cdot \\
&\quad \cdot \Big[s^{(m+n)}(k+h-1, m+h-1) s^{(m+n)}(m+h-1, m+j_{h-1}) \Big] \cdots \\
&\quad \cdots \Big[s^{(m+n)}(k+1, m+1) s^{(m+n)}(m+1, m+j_1) \Big] \cdot \\
&\quad \cdot \Big[s^{(m)}(k, i_k) \otimes 1^{(n)} \Big] \Big[s^{(m)}(k-1, i_{k-1}) \otimes 1^{(n)} \Big] \cdots \Big[s^{(m)}(1, i_1) \otimes 1^{(n)} \Big]
\end{aligned}$$

Now notice that each of the factors

$$s^{(m+n)}(k+h-1,m+h-1),\ s^{(m+n)}(k+h-2,m+h-2),\dots,s^{(m+n)}(k+1,m+1)$$

can be moved to the left past all the factors

$$s^{(m+n)}(m+h,m+j_h),\ s^{(m+n)}(m+h-1,m+j_{h-1}),\dots,s^{(m+n)}(m+2,m+j_2),$$

simply because of (A1): $s^{(m+n)}(k+h-1,m+h-1)$ only involves strands $m+h-1$ and lower, while $s^{(m+n)}(m+h,m+j_h)$ only involves strands $m+h$ and higher; similarly for the others. After performing this commutation we get that

$$\begin{aligned}
s^{(m+n)}_{I\cup(m+J)} &= s^{(m+n)}(k+h,m+h)s^{(m+n)}(k+h-1,m+h-1)\dots s^{(m+n)}(k+1,m+1)\cdot\\
&\cdot s^{(m+n)}(m+h,m+j_h)s^{(m+n)}(m+h-1,m+j_{h-1})\dots s^{(m+n)}(m+1,m+j_1)\cdot\\
&\cdot s^{(m)}_I\otimes 1^{(n)}\\
&\overset{(2),(3)}{=}\Big[1^{(h-1)}\otimes s^{(m+n-h+1)}(k+1,m+1)\Big]\Big[1^{(h-2)}\otimes s^{(m+n-h+1)}(k+1,m+1)\otimes 1\Big]\cdots\\
&\cdots\Big[s^{(m+n-h+1)}(k+1,m+1)\otimes 1^{(h-1)}\Big]\cdot\\
&\cdot\Big[1^{(m)}\otimes s^{(n)}(h,j_h)\Big]\Big[1^{(m)}\otimes s^{(n)}(h-1,j_{h-1})\Big]\cdots\Big[1^{(m)}\otimes s^{(n)}(1,j_1)\Big]\cdot s^{(m)}_I\otimes 1^{(n)}\\
&\overset{(10)}{=}s^{(m+n-h+1)}(k+1,m+1)^{\langle h\rangle}\cdot 1^{(m)}\otimes s^{(n)}_J\cdot s^{(m)}_I\otimes 1^{(n)}\\
&\overset{(2),(3)}{=}\Big[1^{(k)}\otimes s^{(m-k+1)}(1,m-k+1)\otimes 1^{(n-h)}\Big]^{\langle h\rangle}\cdot s^{(m)}_I\otimes s^{(n)}_J\\
&\overset{(11)}{=}1^{(k)}\otimes s^{(m-k+1)}(1,m-k+1)^{\langle h\rangle}\otimes 1^{(n-h)}\cdot s^{(m)}_I\otimes s^{(n)}_J\\
&\overset{(12)}{=}1^{(k)}\otimes\beta_{m-k,h}\otimes 1^{(n-h)}\cdot s^{(m)}_I\otimes s^{(n)}_J\ .
\end{aligned}$$

Thus (*) holds and the proof is complete.

□

B.5 Natural and factorial braids

B.5.1 Definition

The q-analog of a natural number n is

$$[n] = 1 + q + q^2 + \ldots + q^{n-1} \ .$$

For $n \geq 1$, the natural braid $[n] \in kB_n$ is defined as

$$[n] = \sum_{i=1}^{n} s^{(n)}(1,i) = 1 + s_1^{(n)} + s_1^{(n)} s_2^{(n)} + \ldots + s_1^{(n)} s_2^{(n)} \ldots s_{n-1}^{(n)} \ ;$$

we also set $[0] = 0 \in kB_0$.

Notice that $[n] = b_1^{(n)}$. Hence, as a particular case of Vandermonde's formula (15) we have:

$$[m+n] = [m]{\otimes}1^{(n)} + s^{(m+n)}(1, m+1) \cdot 1^{(m)}{\otimes}[n] \ ;$$

since $\beta_{m,1} = s^{(m+1)}(1, m+1)$.

While $[1] = \widetilde{[1]} = [1]^*$ and $[2] = \widetilde{[2]} = [2]^*$, we have

$$[3] = 1 + s_1^{(3)} + s_1^{(3)} s_2^{(3)}, \ \widetilde{[3]} = 1 + s_2^{(3)} + s_2^{(3)} s_1^{(3)} \text{ and } [3]^* = 1 + s_1^{(3)} + s_2^{(3)} s_1^{(3)} \ ;$$

thus $b_i^{(n)^*}$ is not another binomial braid in general. However, it will turn out (18) that the factorial braids are symmetric with respect to both $^{\sim}$ and *.

The q-analog of the factorial number $n!$ is

$$[n]! = \sum_{\sigma \in S_n} q^{\mathsf{inv}(\sigma)} \ ;$$

where the *inversion index* of a permutation $\sigma \in S_n$ is defined as

$$\mathsf{inv}(\sigma) = \#\{(i,j) \;/\; i < j \text{ but } \sigma(i) > \sigma(j)\}.$$

The braid analog of this definition is as follows. First, for any $\sigma \in S_n$ and $i = 1, \dots, n$ let

$$r_i(\sigma) = \#\{j > i \;/\; \sigma(j) < \sigma(i)\}.$$

Thus,

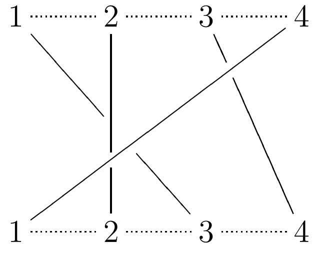
$$\mathsf{inv}(\sigma) = \sum_{i=1}^{n} r_i(\sigma) \ .$$

Notice that $\sigma(i) - i \leq r_i(\sigma) \leq \sigma(i) - 1 \ \forall \ i$, hence it makes sense to define a braid $s_\sigma^{(n)} \in B_n$ as

$$s_\sigma^{(n)} = s^{(n)}(\sigma(n) - r_n(\sigma), n) \cdot \ldots \cdot s^{(n)}(\sigma(2) - r_2(\sigma), 2) \cdot s^{(n)}(\sigma(1) - r_1(\sigma), 1) \ .$$

For instance if $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{smallmatrix}\right)$ then

$$s_\sigma^{(4)} \ =$$



The picture of $s_\sigma^{(n)}$ is obtained by drawing a straight line from 1 in the bottom to $\sigma(1)$ in the top, then *under* that a straight line from 2 to $\sigma(2)$, etc.

In section B.5.3, other expressions for $s_\sigma^{(n)}$ will be given.

Now, for every $n \geq 1$ we define the factorial braid $f^{(n)} \in kB_n$ as

$$f^{(n)} = \sum_{\sigma \in S_n} s_\sigma^{(n)} \ ;$$

we also set $f^{(0)} = 1 \in kB_0$.

We next show that the factorial and natural braids are related by means of a product formula, generalizing $[n]! = [n][n-1]\cdots[2][1]$ for q-numbers. Variations of this will follow after we study the effect of $\,\tilde{}\,$ and * on the $s_\sigma^{(n)}$'s.

Proposition. For every $n \geq 1$,

$$f^{(n)} = 1^{(n-1)}{\otimes}[1] \cdot 1^{(n-2)}{\otimes}[2] \cdot \ldots \cdot 1{\otimes}[n-1] \cdot [n] \ . \tag{16}$$

Proof. We need to show that $f^{(n)} = 1{\otimes}f^{(n-1)} \cdot [n] \ \forall \ n \geq 1$.

Consider the bijection $S_{n-1} \times \{1, 2, \ldots, n\} \to S_n$, $(\sigma, i) \mapsto (1{\otimes}\sigma)(1, 2, \ldots, i)$. (From $\tau := (1{\otimes}\sigma)(1, 2, \ldots, i)$ we recover i as $\tau^{-1}(1)$ and then $1{\otimes}\sigma$ as $\tau{\cdot}(1, 2, \ldots, i)^{-1}$; here $1{\otimes}\sigma$ is such that $(1{\otimes}\sigma)(j) = \sigma(j-1)+1$.) It suffices to show that

$$s_\tau^{(n)} = 1{\otimes}s_\sigma^{(n-1)} \cdot s^{(n)}(1, i) \ .$$

Since $\tau = \left(\begin{smallmatrix} 1 & \ldots & i-1 & i & i+1 & \ldots & n \\ \sigma(1)+1 & \ldots & \sigma(i-1)+1 & 1 & \sigma(i)+1 & \ldots & \sigma(n-1)+1 \end{smallmatrix}\right)$, we have that

$$r_j(\tau) = \begin{cases} r_{j-1}(\sigma) & \text{if } j = i+1, \ldots, n, \\ 0 & \text{if } j = i, \\ r_j(\sigma)+1 & \text{if } j = 1, \ldots, i-1. \end{cases} \ .$$

Hence

$$\begin{aligned}
s_\tau^{(n)} &= s^{(n)}(\tau(n)-r_n(\tau),n)\cdot\ldots\cdot s^{(n)}(\tau(i+1)-r_{i+1}(\tau),i+1)\cdot s^{(n)}(\tau(i)-r_i(\tau),i)\cdot\\
&\quad\cdot s^{(n)}(\tau(i-1)-r_{i-1}(\tau),i-1)\cdot\ldots\cdot s^{(n)}(\tau(1)-r_1(\tau),1)\\
&= s^{(n)}(\sigma(n-1)+1-r_{n-1}(\sigma),n)\cdot\ldots\cdot s^{(n)}(\sigma(i)+1-r_i(\sigma),i+1)\cdot s^{(n)}(1,i)\cdot\\
&\quad\cdot s^{(n)}(\sigma(i-1)+1-r_{i-1}(\sigma)-1,i-1)\cdot\ldots\cdot s^{(n)}(\sigma(1)+1-r_1(\sigma)-1,1)\\
&\overset{(5)}{=} s^{(n)}(\sigma(n-1)-r_{n-1}(\sigma)+1,n-1+1)\cdot\ldots\cdot s^{(n)}(\sigma(i)-r_i(\sigma)+1,i+1)\cdot\\
&\quad\cdot s^{(n)}(\sigma(i-1)-r_{i-1}(\sigma)+1,i-1+1)\cdot\ldots\cdot s^{(n)}(\sigma(1)-r_1(\sigma)+1,1+1)\cdot\\
&\quad\cdot s^{(n)}(1,i)\overset{(3)}{=}1\otimes s_\sigma^{(n-1)}\cdot s^{(n)}(1,i)
\end{aligned}$$

and the proof is complete. □

B.5.2 Symmetries of the factorial braids

To obtain the announced symmetry of the $f^{(n)}$'s, we first describe a multiplicativity property of the map $\xi : S_n \to B_n$, $\sigma \mapsto s_\sigma^{(n)}$. From its definition it is clear that ξ is a section of the canonical projection $B_n \to S_n$, and that $\xi((i,i+1)) = s_i^{(n)}$.[1]

Lemma. Let $\sigma = \sigma_{i_1}\cdot\ldots\cdot\sigma_{i_l} \in S_n$ be a reduced expression for σ as a product of elementary transpositions $\sigma_{i_j} = (i_j, i_j+1)$. Then $s_\sigma^{(n)} = s_{i_1}^{(n)}\cdot\ldots\cdot s_{i_l}^{(n)}$.

Proof. We are given that $\mathsf{length}(\sigma) = l$, where the length of a permutation is the minimum number of elementary transpositions required to write it as a product of such. We will make use of the well-known fact that $\mathsf{inv} = \mathsf{length}$.

---

[1] Lusztig [Lus,2.1.2] has considered sections of this sort for arbitrary Weyl groups W. From lemma (B.5.2) it follows that ξ coincides with Lusztig's section for $W = S_n$.

Clearly, it suffices to show that if $\sigma = \tau \cdot (i, i+1)$ and $\mathsf{length}(\sigma) = \mathsf{length}(\tau) + 1$ then $s_\sigma^{(n)} = s_\tau^{(n)} \cdot s_i^{(n)}$. In this case, $\sigma = \left(\begin{smallmatrix} 1 & \dots & i-1 & i & i+1 & i+2 & \dots & n \\ \tau(1) & \dots & \tau(i-1) & \tau(i+1) & \tau(i) & \tau(i+2) & \dots & \tau(n) \end{smallmatrix}\right)$. Hence $r_j(\sigma) = r_j(\tau)\ \forall\ j \neq i, i+1$. We claim that $\tau(i) < \tau(i+1)$. For if not, we would have $r_i(\sigma) = r_{i+1}(\tau)$ and $r_{i+1}(\sigma) = r_i(\tau) - 1$, from where $\mathsf{length}(\sigma) = \mathsf{inv}(\sigma) = \sum_{j=1}^n r_j(\sigma) = \mathsf{length}(\tau) - 1$, against our hypothesis. Thus $\tau(i) < \tau(i+1)$, and then $r_i(\sigma) = r_{i+1}(\tau) + 1$ and $r_{i+1}(\sigma) = r_i(\tau)$. Hence,

$$
\begin{aligned}
s_\sigma^{(n)} &= s^{(n)}(\sigma(n) - r_n(\sigma), n) \cdot \ldots \cdot s^{(n)}(\sigma(1) - r_1(\sigma), 1) \\
&= s^{(n)}(\tau(n) - r_n(\tau), n) \cdot \ldots \cdot s^{(n)}(\tau(i+2) - r_{i+2}(\tau), i+2) \cdot \\
&\quad s^{(n)}(\tau(i) - r_i(\tau), i+1) \cdot s^{(n)}(\tau(i+1) - r_{i+1}(\tau) - 1, i) \cdot \\
&\quad s^{(n)}(\tau(i-1) - r_{i-1}(\tau), i-1) \cdot \ldots \cdot s^{(n)}(\tau(1) - r_1(\tau), 1) \\
&\overset{(5)}{=} s^{(n)}(\tau(n) - r_n(\tau), n) \cdot \ldots \cdot s^{(n)}(\tau(i+2) - r_{i+2}(\tau), i+2) \cdot \\
&\quad s^{(n)}(\tau(i+1) - r_{i+1}(\tau), i+1) \cdot s^{(n)}(\tau(i) - r_i(\tau), i) \cdot s_i^{(n)} \cdot \\
&\quad \cdot s^{(n)}(\tau(i-1) - r_{i-1}(\tau), i-1) \cdot \ldots \cdot s^{(n)}(\tau(1) - r_1(\tau), 1) \\
&\overset{(A1)}{=} s^{(n)}(\tau(n) - r_n(\tau), n) \cdot \ldots \cdot s^{(n)}(\tau(1) - r_1(\tau), 1) \cdot s_i^{(n)} = s_\tau^{(n)} \cdot s_i^{(n)}
\end{aligned}
$$

and the proof is complete. $\square$

Corollary.

$$\widetilde{s^{(n)}_{\sigma}} = s^{(n)}_{\widetilde{\sigma}},\ \ s^{(n)*}_{\sigma} = s^{(n)}_{\sigma^{-1}}, \quad \text{where } \widetilde{\sigma}(j) = n+1-\sigma(n+1-j) \tag{17}$$

$$f^{(n)} = \widetilde{f^{(n)}} = f^{(n)*} \tag{18}$$

$$\begin{aligned} f^{(n)} &= 1^{(n-1)}\otimes[1]\cdot 1^{(n-2)}\otimes[2]\cdot\ldots\cdot 1\otimes[n-1]\cdot[n] \\ &= \widetilde{[1]}\otimes 1^{(n-1)}\cdot\widetilde{[2]}\otimes 1^{(n-2)}\cdot\ldots\cdot\widetilde{[n-1]}\otimes 1\cdot\widetilde{[n]} \\ &= [n]^{*}\cdot 1\otimes[n-1]^{*}\cdot\ldots\cdot 1^{(n-2)}\otimes[2]^{*}\cdot 1^{(n-1)}\otimes[1]^{*} \\ &= \widetilde{[n]}^{*}\cdot\widetilde{[n-1]}^{*}\otimes 1\cdot\ldots\cdot\widetilde{[2]}^{*}\otimes 1^{(n-2)}\cdot\widetilde{[1]}^{*}\otimes 1^{(n-1)}\ . \end{aligned} \tag{19}$$

Proof. To prove (17), it suffices by the lemma to check these equalities on the elementary transpositions, since both $^{\sim}$ and $^{-1}$ preserve the length of a permutation. But in this case they hold by definition of $^{\sim}$ and * for braids. Then (18) follows by summing over all $\sigma \in S_n$, and the product formulas (19) follow from (16) and (18). □

B.5.3 Other expressions for $s^{(n)}_{\sigma}$.

For any $\sigma \in S_n$ and $i = 1, \ldots, n$ let

$$e_i(\sigma) = \#\{j \leq i \ / \ \sigma(j) \leq \sigma(i)\}.$$

There is a simpler expression for $s^{(n)}_{\sigma}$ in terms of the e_i's.

Proposition. For any $\sigma \in S_n$ and $i = 1, \ldots, n$, $\sigma(i) = r_i(\sigma) + e_i(\sigma)$. Hence

$$s^{(n)}_{\sigma} = s^{(n)}(e_n(\sigma), n)\cdot\ldots\cdot s^{(n)}(e_2(\sigma), 2)\cdot s^{(n)}(e_1(\sigma), 1)\ . \tag{20}$$

Proof.

$$
\begin{aligned}
r_i(\sigma)+e_i(\sigma) &= \#\{j>i\ /\ \sigma(j)<\sigma(i)\}+\#\{j\le i\ /\ \sigma(j)\le\sigma(i)\}\\
&= \#\{j>i\ /\ \sigma(j)\le\sigma(i)\}+\#\{j\le i\ /\ \sigma(j)\le\sigma(i)\}\\
&= \#\{j/\ \sigma(j)\le\sigma(i)\}=\#\{j\ /\ \sigma(j)\in\{1,2,\dots,\sigma(i)\}\ \}\\
&= \sigma(i)\ .
\end{aligned}
$$

$\square$

For completeness, we provide another expression for $s_\sigma^{(n)}$, this time in terms of some partial inversion indices that are obtained by reading σ from right to left. For any $i=1,\dots,n$ let

$$l_i(\sigma)=\#\{j<i\ /\ \sigma(j)>\sigma(i)\}.$$

Proposition. For any $\sigma\in S_n$, $s_\sigma^{(n)}=$

$$s^{(n)}(n,\sigma^{-1}(n)+l_n(\sigma^{-1}))\cdot\ldots\cdot s^{(n)}(2,\sigma^{-1}(2)+l_2(\sigma^{-1}))\cdot s^{(n)}(1,\sigma^{-1}(1)+l_1(\sigma^{-1})).$$

Proof. Notice that $r_i(\sigma)=l_{n+1-i}(\widetilde{\sigma})\ \forall\ i=1,\dots,n$. Hence

$$
\begin{aligned}
s_\sigma^{(n)} &= s^{(n)}(\sigma(n)-r_n(\sigma),n)\cdot\ldots\cdot s^{(n)}(\sigma(1)-r_1(\sigma),1)\\
&= s^{(n)}(\sigma(n)-l_1(\widetilde{\sigma}),n)\cdot\ldots\cdot s^{(n)}(\sigma(1)-l_n(\widetilde{\sigma}),1)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \widetilde{s_\sigma^{(n)}}^{*} &= s^{(n)}(\sigma(\widetilde{1)-l}_n(\widetilde{\sigma}),1)^*\cdot\ldots\cdot s^{(n)}(\sigma(\widetilde{n)-l}_1(\widetilde{\sigma}),n)^*\\
&\overset{(8)}{=} s^{(n)}(n,n+1-\sigma(1)+l_n(\widetilde{\sigma}))\cdot\ldots\cdot s^{(n)}(1,n+1-\sigma(n)+l_1(\widetilde{\sigma}))\\
&= s^{(n)}(n,\widetilde{\sigma}(n)+l_n(\widetilde{\sigma}))\cdot\ldots\cdot s^{(n)}(1,\widetilde{\sigma}(1)+l_1(\widetilde{\sigma}))\\
\overset{(17)}{\Rightarrow} s_{\widetilde{\sigma}^{-1}}^{(n)} &= s^{(n)}(n,\widetilde{\sigma}(n)+l_n(\widetilde{\sigma}))\cdot\ldots\cdot s^{(n)}(1,\widetilde{\sigma}(1)+l_1(\widetilde{\sigma}))\ .
\end{aligned}
$$

Replacing σ by $\widetilde{\sigma}^{-1}$ yields the result. $\square$

B.5.4 Factorial formulas for the binomial coefficients

Next, we present the analog of the well-known formula $\left[{n-i \atop j-i}\right]\left[{n \atop i}\right] = \left[{j \atop i}\right]\left[{n \atop j}\right]$ for q-binomials, from which the factorial formula will be deduced. We choose to provide a bijective proof, even though a proof based on Pascal's identity is possible and shorter, in particular because it yields the stronger result (*) below.

Proposition. Whenever $0 \leq i \leq j \leq n$,

$$1^{(i)} \otimes b_{j-i}^{(n-i)} \cdot b_i^{(n)} = b_i^{(j)} \otimes 1^{(n-j)} \cdot b_j^{(n)} \tag{21}$$

Proof. Consider the map $\mathcal{S}_j(n) \times \mathcal{S}_i(j) \to \mathcal{S}_{j-i}(n-i) \times \mathcal{S}_i(n)$, $(A, B) \mapsto (X, Y)$, defined as follows. First consider the unique order-preserving bijection $k : \{1, \ldots, j\} \to A$ and let $Y = k(B) \in \mathcal{S}_i(n)$, then consider the unique order-preserving bijection $f : \{1, \ldots, n\} \setminus Y \to \{1, \ldots, n-i\}$ and let $X := f(A \setminus Y) \in \mathcal{S}_{j-i}(n-i)$.

Given $(X, Y) \in \mathcal{S}_{j-i}(n-i) \times \mathcal{S}_i(n)$ one recovers $A = Y \cup f^{-1}(X)$ and $B = k^{-1}(Y)$; thus, $(A, B) \to (X, Y)$ is a bijection, so to obtain the result it suffices to prove that

$$1^{(i)} \otimes s_X^{(n-i)} \cdot s_Y^{(n)} = s_B^{(j)} \otimes 1^{(n-j)} \cdot s_A^{(n)} \ . \tag{*}$$

We start by examining the right hand side. Write $A = \{k_1 < \ldots < k_j\} \subseteq \{1, \ldots, n\}$ and $B = \{h_1 < \ldots < h_j\} \subseteq \{1, \ldots, j\}$. Notice that then $Y := \{k_{h_1}, \ldots, k_{h_i}\} \subseteq \{1, \ldots, n\}$.

For each $r = 0, \ldots, i$ let $s_{A_r}^{(n)} := \prod_{h_r < z < h_{r+1}} s^{(n)}(z, k_z)$. (This and all products below are taken in the *decreasing* order: the index z decreases from left to right.

If the interval (h_r, h_{r+1}) is empty then we take $s^{(n)}_{A_r} = 1$; also, we set $h_0 = 0$ and $h_{i+1} = j + 1$.) Then, by definition,

$$s^{(n)}_A = \prod_{0<z<j+1} s^{(n)}(z, k_z) =$$

$$= s^{(n)}_{A_i} \cdot s^{(n)}(h_i, k_{h_i}) \cdot \ldots \cdot s^{(n)}_{A_2} \cdot s^{(n)}(h_2, k_{h_2}) \cdot s^{(n)}_{A_1} \cdot s^{(n)}(h_1, k_{h_1}) \cdot s^{(n)}_{A_0} \ .$$

Hence

$$\begin{aligned} s^{(j)}_B \otimes 1^{(n-j)} \cdot s^{(n)}_A &= s^{(n)}_B s^{(n)}_A \\ &= s^{(n)}(i, h_i) \cdot \ldots \cdot s^{(n)}(2, h_2) \cdot s^{(n)}(1, h_1) \cdot \\ &\cdot s^{(n)}_{A_i} \cdot s^{(n)}(h_i, k_{h_i}) \cdot \ldots \cdot s^{(n)}_{A_2} \cdot s^{(n)}(h_2, k_{h_2}) \cdot s^{(n)}_{A_1} \cdot s^{(n)}(h_1, k_{h_1}) \cdot s^{(n)}_{A_0} \ . \end{aligned}$$

In this expression, $s^{(n)}(1, h_1)$ commutes with all the factors to its right until $s^{(n)}_{A_1}$, including it, since these only involve strands $h_1 + 1$ and higher. When placed there, it joins $s^{(n)}(h_1, k_{h_1})$ to form $s^{(n)}(1, k_{h_1})$, by (1). Similarly $s^{(n)}(2, h_2)$ commutes past $s^{(n)}_{A_2}$ where it joins $s^{(n)}(h_2, k_{h_2})$ to become $s^{(n)}(2, k_{h_2})$, and finally $s^{(n)}(i, h_i)$ and $s^{(n)}(h_i, k_{h_i})$ become $s^{(n)}(i, k_{h_i})$. After this transformation we get

$$s^{(j)}_B \otimes 1^{(n-j)} \cdot s^{(n)}_A = s^{(n)}_{A_i} \cdot s^{(n)}(i, k_{h_i}) \cdot \ldots \cdot s^{(n)}_{A_2} \cdot s^{(n)}(2, k_{h_2}) \cdot s^{(n)}_{A_1} \cdot s^{(n)}(1, k_{h_1}) \cdot s^{(n)}_{A_0} \ .$$

Now notice that each factor in $s^{(n)}_{A_0}$ is of the form $s^{(n)}(z, k_z)$ with $1 \leq z < h_1$, hence by (5) and (3)

$$s^{(n)}(1, k_{h_1}) \cdot s^{(n)}_{A_0} = 1 \otimes s^{(n-1)}_{A_0} \cdot s^{(n)}(1, k_{h_1}) \ .$$

Similarly we can now commute $s^{(n)}_{A_1} \cdot 1 \otimes s^{(n-1)}_{A_0}$ past $s^{(n)}(2, k_{h_2})$, using (5) and (3); this factor becomes $1 \otimes s^{(n)}_{A_1} \cdot 1^{(2)} \otimes s^{(n-1)}_{A_0}$ when placed to the left of $s^{(n)}(2, k_{h_2})$. After doing this for each $r = 0, \ldots, i-1$ we get

$$\begin{aligned} s_B^{(j)}\otimes 1^{(n-j)} \cdot s_A^{(n)} &= s_{A_i}^{(n)} \cdot 1\otimes s_{A_{i-1}}^{(n-1)} \cdot \ldots \cdot 1^{(i-2)}\otimes s_{A_2}^{(n)} \cdot 1^{(i-1)}\otimes s_{A_1}^{(n)} \cdot 1^{(i)}\otimes s_{A_0}^{(n)} \cdot \\ &\quad \cdot s^{(n)}(i, k_{h_i}) \cdot \ldots \cdot s^{(n)}(2, k_{h_2}) \cdot s^{(n)}(1, k_{h_1}) \\ &= \prod_{r=0}^{i} 1^{(i-r)}\otimes s_{A_r}^{(n-i+r)} \cdot s_Y^{(n)} \ . \end{aligned}$$

Thus, to obtain (*), we need to show that

$$1^{(i)}\otimes s_X^{(n-i)} = \prod_{r=0}^{i} 1^{(i-r)}\otimes s_{A_r}^{(n-i+r)} \tag{**}$$

To this end, we describe f and X explicitly. By definition, $f : \{1, \ldots, n\} \setminus \{k_{h_1}, \ldots k_{h_i}\} \to \{1, \ldots, n-i\}$ is translation by $-r$ on each open interval $(k_{h_r}, k_{h_{r+1}})$, for $r = 0, \ldots, i$ (where we set $k_0 = 0$ and $k_{j+1} = n+1$). Then, since

$$A \setminus Y = k(\{1, \ldots, j\} \setminus \{h_1, \ldots, h_i\}) = \bigcup_{r=0}^{i} k((h_r, h_{r+1})) \ ,$$

we have that

$$X = f(A \setminus Y) = \bigcup_{r=0}^{i} k((h_r, h_{r+1})) - r \ .$$

Thus, letting $s_{X_r}^{(n-i)} := \prod_{h_r < z < h_{r+1}} s^{(n-i)}(z - r, k_z - r)$, we have that $s_X^{(n-i)} = \prod_{r=0}^{i} s_{X_r}^{(n-i)}$. But notice that

$$\begin{aligned} 1^{(i)}\otimes s_{X_r}^{(n-i)} &= \prod_{h_r < z < h_{r+1}} 1^{(i)}\otimes s^{(n-i)}(z - r, k_z - r) \overset{(3)}{=} \prod_{h_r < z < h_{r+1}} s^{(n)}(z + i - r, k_z + i - r) \\ &\overset{(3)}{=} 1^{(i-r)}\otimes \prod_{h_r < z < h_{r+1}} s^{(n-i+r)}(z, k_z) = 1^{(i-r)}\otimes s_{A_r}^{(n-i+r)} \ , \end{aligned}$$

hence

$$1^{(i)}\otimes s_X^{(n-i)} = \prod_{r=0}^{i} 1^{(i)}\otimes s_{X_r}^{(n-i)} = \prod_{r=0}^{i} 1^{(i-r)}\otimes s_{A_r}^{(n-i+r)}$$

so (**) holds and the proof is complete. □

We can now derive the braid analog of the usual expression for the binomial coefficients in terms of factorials.

Corollary. Whenever $0 \leq j \leq n$,

$$f^{(j)} \otimes f^{(n-j)} \cdot b_j^{(n)} = f^{(n)} \ . \tag{22}$$

Proof. Formula (21) with $i = 1$ says

$$1 \otimes b_{j-1}^{(n-1)} \cdot [n] = [j] \otimes 1^{(n-j)} \cdot b_j^{(n)} \ .$$

Repeated use of this yields

$$\begin{aligned} &1^{(j-1)} \otimes [n-j+1] \cdot 1^{(j-2)} \otimes [n-j+2] \cdot \ldots \cdot 1 \otimes [n-1] \cdot [n] \\ &\quad = 1^{(j-1)} \otimes [1] \otimes 1^{(n-j)} \cdot 1^{(j-2)} \otimes [2] \otimes 1^{(n-j)} \cdot \ldots \cdot 1 \otimes [j-1] \otimes 1^{(n-j)} \cdot [j] \otimes 1^{(n-j)} \cdot b_j^{(n)} \\ &\quad \overset{(16)}{=} f^{(j)} \otimes 1^{(n-j)} \cdot b_j^{(n)} \end{aligned}$$

Multiplying both sides by $1^{(j)} \otimes f^{(n-j)}$ and using (16) we get the result. ☐

It seems that in the course of the proof of (22) we obtained a stronger "simplified" formula; in fact this is equivalent to (22) since the braid group algebras do not possess zero divisors [2].

Recall that the natural braids $[j]$ are not $\widetilde{\ }$-symmetric. However, an amusing consequence of (21) is this (choosing $n = j+1$, $i = 1$):

$$1 \otimes \widetilde{[j]} \cdot [j+1] = [j] \otimes 1 \cdot \widetilde{[j+1]} \ .$$

Thus this element is fixed by $\widetilde{\ }$.

---

[2] In fact, B_n is *right-ordered* by a recent result of Dehornoy [Deh], hence kB_n does not possess zero divisors nor non-trivial units by the results in chapter 13.1 of Passman's book [Pas]. We thank Dale Rolfsen for making us aware of this.

B.6 Rota's binomial theorem, Cauchy's identities and Möbius inversion

B.6.1 The binomial theorem

The following remarkable q-binomial theorem is proven in [GR1]: if $P_k(\mathbf{x},\mathbf{y}) = (\mathbf{x}-\mathbf{y})(\mathbf{x}-q\mathbf{y})\dots(\mathbf{x}-q^{k-1}\mathbf{y})$ then

$$P_n(\mathbf{x},\mathbf{z}) = \sum_{k=0}^{n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] P_k(\mathbf{x},\mathbf{y}) P_{n-k}(\mathbf{y},\mathbf{z}) \ ,$$

this is an identity in the ordinary polynomial ring $k[\mathbf{x},\mathbf{y},\mathbf{z}]$. When $q = 1$ this reduces to the familiar

$$(\mathbf{x}-\mathbf{z})^n = \sum_{k=0}^{n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] (\mathbf{x}-\mathbf{y})^k (\mathbf{y}-\mathbf{z})^{n-k} \ .$$

We will generalize this result to the context of braids, and derive from it the other results of the section.

We consider ordinary polynomial rings $kB_n[\mathbf{x}_1,\dots,\mathbf{x}_r]$ over the non-commutative ring kB_n; thus, the variables commute among themselves and with the coefficients. The embeddings

$$B_k \to B_n,\ s \mapsto s{\scriptstyle\otimes}1^{(n-k)} \ \text{ and } \ B_{n-k} \to B_n,\ t \mapsto 1^{(k)}{\scriptstyle\otimes}t$$

extend to embeddings

$$\begin{array}{rcl} kB_k[\mathbf{x}_1,\dots,\mathbf{x}_r] & \to & kB_n[\mathbf{x}_1,\dots,\mathbf{x}_r] \\ p & \mapsto & p{\scriptstyle\otimes}1^{(n-k)} \end{array} \ \text{ and } \ \begin{array}{rcl} kB_{n-k}[\mathbf{x}_1,\dots,\mathbf{x}_r] & \to & kB_n[\mathbf{x}_1,\dots,\mathbf{x}_r], \\ q & \mapsto & 1^{(k)}{\scriptstyle\otimes}q \end{array}$$

where $\mathbf{x}_i$ is sent to $\mathbf{x}_i$ in both cases. The images of these maps commute elementwise inside $kB_n[\mathbf{x}_1, \ldots , \mathbf{x}_r]$, so there is an induced map

$$kB_k[\mathbf{x}_1, \ldots , \mathbf{x}_r]{\otimes}kB_{n-k}[\mathbf{x}_1, \ldots , \mathbf{x}_r] \to kB_n[\mathbf{x}_1, \ldots , \mathbf{x}_r],\ p{\otimes}q \mapsto p{\otimes}1^{(n-k)} \cdot 1^{(k)}{\otimes}q\ .$$

We will write $p{\otimes}q$ for $p{\otimes}1^{(n-k)} \cdot 1^{(k)}{\otimes}q$.

For any $k \geq 1$ let

$$P_k(\mathbf{x}, \mathbf{y}) = [\mathbf{x} - s^{(k)}(1, k)\mathbf{y}] \cdot [\mathbf{x} - s^{(k)}(1, k-1)\mathbf{y}] \cdot \ldots \cdot [\mathbf{x} - s^{(k)}(1, 1)\mathbf{y}] \ \in kB_k[\mathbf{x}, \mathbf{y}]\ ;$$

and set $P_0(\mathbf{x}, \mathbf{y}) = 1 \in kB_0$.

Then, with the above convention, the binomial theorem is the following identity in $kB_n[\mathbf{x}, \mathbf{y}, \mathbf{z}]$:

Proposition. For any $n \geq 0$,

$$P_n(\mathbf{x}, \mathbf{z}) = \sum_{k=0}^{n} P_k(\mathbf{y}, \mathbf{z}){\otimes}P_{n-k}(\mathbf{x}, \mathbf{y}) \cdot b_k^{(n)}\ . \tag{23}$$

Proof. We do induction on n. For $n = 0, 1$ the statement is trivial. Assuming it true for $n-1$ with $n \geq 2$, we have

$$P_n(\mathbf{x},\mathbf{z}) = [\mathbf{x} - s^{(n)}(1,n)\mathbf{z}] \cdot [P_{n-1}(\mathbf{x},\mathbf{z}) {\otimes} 1]$$

$$= [\mathbf{x} - s^{(n)}(1,n)\mathbf{z}] \cdot \sum_{k=0}^{n-1} [P_k(\mathbf{y},\mathbf{z}) {\otimes} P_{n-1-k}(\mathbf{x},\mathbf{y}) {\otimes} 1] \cdot [b_k^{(n-1)} {\otimes} 1]$$

$$= \sum_{k=0}^{n-1} [\mathbf{x} - s^{(n)}(k+1,n)\mathbf{y} + s^{(n)}(k+1,n)\mathbf{y} - s^{(n)}(1,n)\mathbf{z}] \cdot [P_k(\mathbf{y},\mathbf{z}) {\otimes} P_{n-1-k}(\mathbf{x},\mathbf{y}) {\otimes} 1] \cdot$$

$$\cdot [b_k^{(n-1)} {\otimes} 1]$$

$$\overset{(A1),\,(1)}{=} \sum_{k=0}^{n-1} [P_k(\mathbf{y},\mathbf{z}) {\otimes} 1^{(n-k)}] \cdot [\mathbf{x} - s^{(n)}(k+1,n)\mathbf{y}] \cdot [1^{(k)} {\otimes} P_{n-1-k}(\mathbf{x},\mathbf{y}) {\otimes} 1] \cdot$$

$$\cdot [b_k^{(n-1)} {\otimes} 1] +$$

$$+ \sum_{k=0}^{n-1} [\mathbf{y} - s^{(n)}(1,k+1)\mathbf{z}] \cdot [P_k(\mathbf{y},\mathbf{z}) {\otimes} 1^{(n-k)}] \cdot s^{(n)}(k+1,n) \cdot [1^{(k)} {\otimes} P_{n-1-k}(\mathbf{x},\mathbf{y}) {\otimes} 1] \cdot$$

$$\cdot [b_k^{(n-1)} {\otimes} 1]$$

$$\overset{(3)}{=} \sum_{k=0}^{n-1} [P_k(\mathbf{y},\mathbf{z}) {\otimes} 1^{(n-k)}] \cdot \Big[1^{(k)} {\otimes} [\mathbf{x} - s^{(n)}(1,n-k)\mathbf{y}][P_{n-1-k}(\mathbf{x},\mathbf{y}) {\otimes} 1]\Big] \cdot [b_k^{(n-1)} {\otimes} 1] +$$

$$+ \sum_{k=0}^{n-1} \Big[[\mathbf{y} - s^{(k+1)}(1,k+1)\mathbf{z}] {\otimes} 1^{(n-k-1)}\Big] \cdot [P_k(\mathbf{y},\mathbf{z}) {\otimes} 1^{(n-k)}] \cdot s^{(n)}(k+1,n) \cdot$$

$$\cdot [1^{(k)} {\otimes} P_{n-1-k}(\mathbf{x},\mathbf{y}) {\otimes} 1] \cdot [b_k^{(n-1)} {\otimes} 1]$$

$$= \sum_{k=0}^{n-1} [P_k(\mathbf{y},\mathbf{z}) {\otimes} 1^{(n-k)}] \cdot [1^{(k)} {\otimes} P_{n-k}(\mathbf{x},\mathbf{y})] \cdot [b_k^{(n-1)} {\otimes} 1] +$$

$$+ \sum_{k=0}^{n-1} [P_{k+1}(\mathbf{y},\mathbf{z}) {\otimes} 1^{(n-k-1)}] \cdot s^{(n)}(k+1,n) \cdot [1^{(k)} {\otimes} P_{n-1-k}(\mathbf{x},\mathbf{y}) {\otimes} 1] \cdot [b_k^{(n-1)} {\otimes} 1] \; .$$

Now we use (5) to commute $s^{(n)}(k+1,n)$ past $P_{n-1-k}(\mathbf{x},\mathbf{y})$ as follows:

$$\begin{aligned}
&s^{(n)}(k+1,n)\cdot[1^{(k)}{\otimes}P_{n-1-k}(\mathbf{x},\mathbf{y}){\otimes}1]\\
&= s^{(n)}(k+1,n)\cdot\Big[1^{(k)}{\otimes}[\mathbf{x}-s^{(n-1-k)}(1,n-1-k)\mathbf{y}]\cdot\ldots\cdot[\mathbf{x}-s^{(n-1-k)}(1,1)\mathbf{y}]{\otimes}1\Big]\\
&\overset{(2),\,(3)}{=} s^{(n)}(k+1,n)\cdot[\mathbf{x}-s^{(n)}(k+1,n-1)\mathbf{y}]\cdot\ldots\cdot[\mathbf{x}-s^{(n)}(k+1,k+1)\mathbf{y}]\\
&\overset{(5)}{=}[\mathbf{x}-s^{(n)}(k+2,n)\mathbf{y}]\cdot\ldots\cdot[\mathbf{x}-s^{(n)}(k+2,k+2)\mathbf{y}]\cdot s^{(n)}(k+1,n)\\
&\overset{(3)}{=}\Big[1^{(k+1)}{\otimes}[\mathbf{x}-s^{(n-1-k)}(1,n-k-1)\mathbf{y}]\cdot\ldots\cdot[\mathbf{x}-s^{(n-k-1)}(1,1)\mathbf{y}]\Big]\cdot s^{(n)}(k+1,n)\\
&=[1^{(k+1)}{\otimes}P_{n-k-1}(\mathbf{x},\mathbf{y})]\cdot s^{(n)}(k+1,n)\ .
\end{aligned}$$

Substituting this in the above expression for P_n we get

$$\begin{aligned}
&P_n(\mathbf{x},\mathbf{z})=\\
&=\sum_{k=0}^{n-1}[P_k(\mathbf{y},\mathbf{z}){\otimes}1^{(n-k)}]\cdot[1^{(k)}{\otimes}P_{n-k}(\mathbf{x},\mathbf{y})]\cdot[b_k^{(n-1)}{\otimes}1]+\\
&\quad+\sum_{k=0}^{n-1}[P_{k+1}(\mathbf{y},\mathbf{z}){\otimes}1^{(n-k-1)}]\cdot[1^{(k+1)}{\otimes}P_{n-1-k}(\mathbf{x},\mathbf{y})]\cdot s^{(n)}(k+1,n)\cdot[b_k^{(n-1)}{\otimes}1]\\
&=\sum_{k=0}^{n-1}[P_k(\mathbf{y},\mathbf{z}){\otimes}P_{n-k}(\mathbf{x},\mathbf{y})]\cdot[b_k^{(n-1)}{\otimes}1]+\\
&\quad+\sum_{k=1}^{n}[P_k(\mathbf{y},\mathbf{z}){\otimes}P_{n-k}(\mathbf{x},\mathbf{y})]\cdot s^{(n)}(k+1,n)\cdot[b_k^{(n-1)}{\otimes}1]\\
&=\sum_{k=0}^{n}[P_k(\mathbf{y},\mathbf{z}){\otimes}P_{n-k}(\mathbf{x},\mathbf{y})]\cdot[b_k^{(n-1)}{\otimes}1+s^{(n)}(k+1,n)\cdot b_k^{(n-1)}{\otimes}1]\\
&\overset{(14)}{=}\sum_{k=0}^{n}[P_k(\mathbf{y},\mathbf{z}){\otimes}P_{n-k}(\mathbf{x},\mathbf{y})]\cdot b_k^{(n)}\ .
\end{aligned}$$

□

B.6.2 Cauchy's identities

These identities are attributed to Cauchy in [GR1]:

$$(\mathbf{x}-1)(\mathbf{x}-q)\dots(\mathbf{x}-q^{n-1}) = \sum_{k=0}^{n} \left[{n \atop k}\right] (-1)^k q^{\binom{k}{2}} \mathbf{x}^{n-k} \ ,$$

$$\mathbf{x}^n = \sum_{k=0}^{n} \left[{n \atop k}\right] (\mathbf{x}-1)(\mathbf{x}-q)\dots(\mathbf{x}-q^{k-1}) \ .$$

Just as in the q-case, its generalizations to braids are easily obtained from the binomial theorem. In this context, it is natural to introduce the *Möbius* braid $\mu^{(k)} \in kB_k$ as

$$\mu^{(k)} = (-1)^k c^{(k)}$$

where $c^{(k)} = s^{(k)}(1,k)s^{(k)}(1,k-1)\dots s^{(k)}(1,1) \in kB_k$ is the twistor braid of section B.2.2.

Corollary. For any $n \geq 0$,

$$[\mathbf{x}-s^{(n)}(1,n)]\cdot[\mathbf{x}-s^{(n)}(1,n-1)]\cdot\ldots\cdot[\mathbf{x}-s^{(n)}(1,1)] = \sum_{k=0}^{n} \mu^{(k)}{\otimes}1^{(n-k)} \cdot b_k^{(n)} \cdot \mathbf{x}^{n-k} \tag{24}$$

$$\mathbf{x}^n = \sum_{k=0}^{n} [\mathbf{x}-s^{(n)}(k+1,n)]\cdot[\mathbf{x}-s^{(n)}(k+1,n-1)]\cdot\ldots\cdot[\mathbf{x}-s^{(n)}(k+1,k+1)]\cdot b_k^{(n)}. \tag{25}$$

Proof. Setting $\mathbf{y}=0$ and $\mathbf{z}=1$ in (23) we obtain (24); setting $\mathbf{y}=1$ and $\mathbf{z}=0$ we obtain (25). These evaluations are well-defined morphisms of algebras because the evaluating points commute with the coefficients. □

Möbius inversion formula will we deduced from the following two consequences of Cauchy's identities. Setting $\mathbf{x} = 1$ in (24) we obtain

$$\sum_{k=0}^{n} \mu^{(k)} \otimes 1^{(n-k)} \cdot b_k^{(n)} = 0 \ \forall \ n > 0, \tag{26}$$

and setting $\mathbf{x} = 0$ in (25) (or applying $\tilde{}$ to (26))

$$\sum_{k=0}^{n} 1^{(k)} \otimes \mu^{(n-k)} \cdot b_k^{(n)} = 0 \ \forall \ n > 0. \tag{27}$$

Both of these reduce in the q-case to the well-known

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \left[{n \atop k} \right] = 0 \ \forall \ n > 0.$$

Some other interesting consequences of Cauchy's identities are obtained through other evaluations; these all reduce to the same identity in the q-case, but are distinct at the level of braids. To briefly discuss this situation, consider the polynomial ring $B[\mathbf{x}]$ over a non-commutative ring B. For each $b \in B$ there are two natural *evaluation* maps $B[\mathbf{x}] \to B$, according to whether we write the variable to the right or left of the coefficients. More precisely, these are defined as

$$\epsilon_b^r : B[\mathbf{x}] \to B, \qquad a_n\mathbf{x}^n + \ldots + a_1\mathbf{x} + a_0 \mapsto a_n b^n + \ldots + a_1 b + a_0$$

and

$$\epsilon_b^l : B[\mathbf{x}] \to B, \qquad a_n\mathbf{x}^n + \ldots + a_1\mathbf{x} + a_0 \mapsto b^n a_n + \ldots + b a_1 + a_0 \ .$$

These maps are not multiplicative in general; however, if h, f and g are polynomials such that $h = fg$ and b commutes with the coefficients of g, then $\epsilon_b^r(h) = \epsilon_b^r(f)\epsilon_b^r(g)$. Similarly, if b commutes with the coefficients of f then $\epsilon_b^l(h) = \epsilon_b^l(f)\epsilon_b^l(g)$.

Consider $B = kB_n$, $f(\mathbf{x}) = [\mathbf{x} - s^{(n)}(1,n)] \cdot [\mathbf{x} - s^{(n)}(1,n-1)] \cdot \ldots \cdot [\mathbf{x} - s^{(n)}(1,3)]$ and $g(\mathbf{x}) = [\mathbf{x} - s^{(n)}(1,2)][\mathbf{x} - s^{(n)}(1,1)]$. Writing $\mathbf{x}$ to the right of the coefficients and evaluating (24) at $b = s^{(n)}(1,2) = s_1^{(n)}$ we obtain

$$\sum_{k=0}^{n} \mu^{(k)}{\otimes}1^{(n-k)} \cdot b_k^{(n)} \cdot (s_1^{(n)})^{n-k} = 0 \ .$$

Similarly, letting $f(\mathbf{x}) = x - s^{(n)}(1,n)$, $g(\mathbf{x}) = [\mathbf{x} - s^{(n)}(1,n-1)] \cdot \ldots \cdot [\mathbf{x} - s^{(n)}(1,1)]$, writing $\mathbf{x}$ to the left and evaluating (24) at $b = s^{(n)}(1,n)$ we obtain

$$\sum_{k=0}^{n} \Big[s^{(n)}(1,n) \Big]^{n-k} \cdot \mu^{(k)}{\otimes}1^{(n-k)} \cdot b_k^{(n)} = 0 \ .$$

B.6.3 Möbius inversion

A particular case of the general theory of Möbius inversion [Rot] is the following q-numerical inversion formula: for any scalars $a_0, \ldots, a_m$, $b_0, \ldots, b_m$,

$$b_i = \sum_{j=0}^{i} \left[{i \atop j} \right] a_{i-j} \ \forall \ i = 0, \ldots, m \iff a_i = \sum_{j=0}^{i} (-1)^j q^{\binom{j}{2}} \left[{i \atop j} \right] b_{i-j} \ \forall \ i = 0, \ldots, m \ .$$

Its generalization is:

Proposition. Let $x^{(i)}$ and $y^{(i)} \in kB_i$ be given braids for $i = 0, \ldots, m$. Then

$$x^{(i)} = \sum_{j=0}^{i} 1^{(j)}{\otimes}y^{(i-j)} \cdot b_j^{(i)} \ \forall \ i = 0 \ldots m \iff y^{(i)} = \sum_{j=0}^{i} \mu^{(j)}{\otimes}x^{(i-j)} \cdot b_j^{(i)} \ \forall \ i = 0 \ldots m \ . \tag{28}$$

Proof. (⇒)

$$
\begin{aligned}
\sum_{j=0}^{i}\mu^{(j)}\otimes x^{(i-j)}\cdot b_j^{(i)} &\overset{(hyp.)}{=} \sum_{j=0}^{i}\mu^{(j)}\otimes\Big[\sum_{h=0}^{i-j}1^{(h)}\otimes y^{(i-j-h)}\cdot b_h^{(i-j)}\Big]\cdot b_j^{(i)}\\
&= \sum_{j=0}^{i}\sum_{h=0}^{i-j}\Big[\mu^{(j)}\otimes 1^{(h)}\otimes y^{(i-j-h)}\Big]\cdot\Big[1^{(j)}\otimes b_h^{(i-j)}\Big]\cdot b_j^{(i)}\\
&\overset{(21)}{=} \sum_{j=0}^{i}\sum_{h=0}^{i-j}\Big[\mu^{(j)}\otimes 1^{(h)}\otimes y^{(i-j-h)}\Big]\cdot\Big[b_j^{(h+j)}\otimes 1^{(i-j-h)}\Big]\cdot b_{h+j}^{(i)}\\
&= \sum_{j=0}^{i}\sum_{h=0}^{i-j}\Big[\mu^{(j)}\otimes 1^{(i-j)}\Big]\cdot\Big[b_j^{(h+j)}\otimes y^{(i-j-h)}\Big]\cdot b_{h+j}^{(i)}\\
&\overset{(k:=h+j)}{=} \sum_{k=0}^{i}\sum_{j=0}^{k}\Big[\mu^{(j)}\otimes 1^{(i-j)}\Big]\cdot\Big[b_j^{(k)}\otimes y^{(i-k)}\Big]\cdot b_k^{(i)} = y^{(i)},
\end{aligned}
$$

since by (24) all terms corresponding to $k \neq 0$ in the above sum vanish.

(⇐)

$$
\begin{aligned}
\sum_{j=0}^{i}1^{(j)}\otimes y^{(i-j)}\cdot b_j^{(i)} &\overset{(hyp.)}{=} \sum_{j=0}^{i}1^{(j)}\otimes\Big[\sum_{h=0}^{i-j}\mu^{(h)}\otimes x^{(i-j-h)}\cdot b_h^{(i-j)}\Big]\cdot b_j^{(i)}\\
&= \sum_{j=0}^{i}\sum_{h=0}^{i-j}\Big[1^{(j)}\otimes\mu^{(h)}\otimes x^{(i-j-h)}\Big]\cdot\Big[1^{(j)}\otimes b_h^{(i-j)}\Big]\cdot b_j^{(i)}\\
&\overset{(21)}{=} \sum_{j=0}^{i}\sum_{h=0}^{i-j}\Big[1^{(j)}\otimes\mu^{(h)}\otimes x^{(i-j-h)}\Big]\cdot\Big[b_j^{(h+j)}\otimes 1^{(i-j-h)}\Big]\cdot b_{h+j}^{(i)}\\
&= \sum_{j=0}^{i}\sum_{h=0}^{i-j}\Big[\Big[1^{(j)}\otimes\mu^{(h)}\Big]\cdot b_j^{(h+j)}\otimes x^{(i-j-h)}\Big]\cdot b_{h+j}^{(i)}\\
&\overset{(k:=h+j)}{=} \sum_{k=0}^{i}\Big[\sum_{j=0}^{k}\Big[1^{(j)}\otimes\mu^{(k-j)}\cdot b_j^{(k)}\Big]\otimes x^{(i-k)}\Big]\cdot b_k^{(i)} = x^{(i)},
\end{aligned}
$$

since by (25) all terms corresponding to $k \neq 0$ in the above sum vanish. □

B.7 Multinomial braids

B.7.1 Definition

For each n and $r \in \mathbb{N}$ let $\mathcal{F}(n,r)$ denote the set of all functions $\{1,\ldots,n\} \to \{1,\ldots,r\}$, and $\mathcal{C}(n,r) = \{(\eta_1,\ldots,\eta_r) \in \mathbb{N}^r \;/\; \eta_1 + \ldots + \eta_r = n\}$. A sequence $\eta \in \mathcal{C}(n,r)$ is sometimes called a *weak composition of n into r parts.* For any $\eta \in \mathcal{C}(n,r)$ let

$$\mathcal{S}(\eta) = \{f \in \mathcal{F}(n,r) \;/\; \#f^{-1}(1) = \eta_1, \#f^{-1}(2) = \eta_2, \ldots, \#f^{-1}(r) = \eta_r\}.$$

We usually write $f = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 1 & 3 & 2 \end{smallmatrix}\right)$ to abbreviate that $f : \{1,2,3,4,5\} \to \{1,2,3\}$ is $f(1) = f(5) = 2$, $f(2) = f(3) = 1$, $f(4) = 3$. One may think of the elements of $\mathcal{S}(\eta)$ as permutations of the elements of $\{1,2,\ldots,r\}$ with repetitions as specified by η. For this reason the elements of $\mathcal{S}(\eta)$ are called *permutations of the multiset* $\{1^{\eta_1}, 2^{\eta_2}, \ldots, r^{\eta_r}\}$.

There are canonical identifications $\mathcal{S}(1,1,\ldots,1) = S_r$ (r ones) and (when $r = 2$) $\mathcal{S}(i, n-i) = \mathcal{S}_i(n)$, $f \mapsto \{j \in \{1,2,\ldots,n\} \;/\; f(j) = 1\}$.

Given $\eta \in \mathcal{C}(n,r)$, the corresponding q-multinomial coefficient is defined as

$$\left[\begin{smallmatrix} n \\ \eta \end{smallmatrix}\right] = \sum_{f \in \mathcal{S}(\eta)} q^{\mathsf{inv}(f)} ,$$

where the *inversion index* $\mathsf{inv}(f)$ is

$$\mathsf{inv}(f) = \#\{(i,j) \;/\; 1 \leq i < j \leq n,\ f(i) > f(j)\} .$$

To define its braid analog we proceed as follows. First, for any $f \in \mathcal{F}(n,r)$ and

$i \in \{1, 2, \dots, n\}$, set

$$e_i(f) = \#\{j \leq i \ / \ f(j) \leq f(i)\} \ .$$

Next, define $s_f^{(n)} \in B_n$ as

$$s_f^{(n)} = s^{(n)}(e_n(f), n) \cdot \ldots \cdot s^{(n)}(e_2(f), 2) \cdot s^{(n)}(e_1(f), 1) \ .$$

Then, for any $\eta \in \mathcal{C}(n, r)$, define the multinomial braid $m^{(\eta)} \in kB_n$ as

$$m^{(\eta)} = \sum_{f \in \mathcal{S}(\eta)} s_f^{(n)} \ ;$$

and $m^{(0,\dots,0)} = 1 \in kB_0$.

A few remarks are in order. First, notice that for $\sigma \in S_r = \mathcal{S}(1, 1, \dots, 1)$ (r ones), the definition of $s_\sigma^{(n)}$ given here coincides with that of section B.5, because of equation (20). Hence $m^{(1,1,\dots,1)} = f^{(r)}$, the factorial braid.

Second, suppose $r = 2$, and let $I \in \mathcal{S}_i(n)$ correspond to $f \in \mathcal{S}(i, n-i)$ under the bijection described above: if $I = \{j_1 < j_2 < \ldots < j_i\}$ then $f = \left(\begin{smallmatrix} 1 & \dots & & j_1 & & \dots & & j_2 & & \dots & & j_i & & \dots & n \\ 2 & \dots & 2 & 1 & 2 & \dots & 2 & 1 & 2 & \dots & 2 & 1 & 2 & \dots & 2 \end{smallmatrix}\right)$.

Thus, $e_j(f) = \begin{cases} j & \text{if } j \notin I, \\ h & \text{if } j = j_h \in I \end{cases}$, from where $s_f^{(n)} = s^{(n)}(i, j_i) \cdot \ldots \cdot s^{(n)}(2, j_2) \cdot s^{(n)}(1, j_1) = s_I^{(n)}$, and hence $m^{(i,n-i)} = b_i^{(n)}$. Thus multinomial braids reduce to binomial braids when $r = 2$.

Finally, let us check that in the one-dimensional representation defined by q (section B.2.5), $s_f^{(n)}$ acts as multiplication by $q^{\mathsf{inv}(f)}$, and hence $m^{(\eta)}$ as $\left[\begin{smallmatrix} n \\ \eta \end{smallmatrix}\right]$.

To this end, we introduce the *η-shuffle* $\sigma_f \in S_n$ corresponding to $f \in \mathcal{S}(\eta)$ as follows: on $f^{-1}(1)$, σ_f is the unique increasing bijection onto $\{1, \dots, \eta_1\}$, similarly on $f^{-1}(2)$ onto $\{\eta_1+1,\ldots,\eta_1+\eta_2\}$, …, and on $f^{-1}(r)$ onto $\{\eta_1+\ldots+\eta_{r-1}+1,\ldots,\eta_1+\ldots+\eta_r\}$.

We also introduce the partial inversion index $r_i(f)=\#\{j>i\ /\ f(j)<f(i)\}$, extending the one already defined for permutations in section B.5. Notice that $\mathsf{inv}(f)=\sum_{i=1}^n r_i(f)$.

Lemma. For any $f\in\mathcal{S}(\eta)$ and $i\in\{1,2,\ldots,n\}$, $e_i(f)=e_i(\sigma_f)$ and $r_i(f)=r_i(\sigma_f)$.

Proof. From the definition of σ_f we see that:

For $j\leq i$, $\sigma_f(j)\leq\sigma_f(i)\Leftrightarrow f(j)\leq f(i)$. From here, $e_i(f)=e_i(\sigma_f)$.

For $j>i$, $\sigma_f(j)<\sigma_f(i)\Leftrightarrow f(j)<f(i)$. From here, $r_i(f)=r_i(\sigma_f)$. $\square$

Now we can show that $s_f^{(n)}$ acts as $q^{\mathsf{inv}(f)}$, i.e. that the number of elementary generators in $s^{(n)}(e_n(f),n)\cdot\ldots\cdot s^{(n)}(e_2(f),2)\cdot s^{(n)}(e_1(f),1)$ is $\mathsf{inv}(f)$. Recall (section B.5.3) that for any $\sigma\in S_n$ we have $\sigma(i)=r_i(\sigma)+e_i(\sigma)$. Hence, $\sigma_f(i)=r_i(\sigma_f)+e_i(\sigma_f)=r_i(f)+e_i(f)$, from where

$$\#\text{generators in } s_f^{(n)}=\sum_{i=1}^n i-e_i(f)=\sum_{i=1}^n\sigma_f(i)-e_i(f)=\sum_{i=1}^n r_i(f)=\mathsf{inv}(f),$$

as needed.

From the lemma we also deduce that $s_f^{(n)}=s_{\sigma_f}^{(n)}$, just comparing their definitions. This shows that our multinomial braids coincide with those braids already considered by Schauenburg in [Sch, definition 2.6]. Some of the identities we prove here ((14), (22), and a particular case of (30)) are stated in that paper, altough the connection to combinatorics is not pointed out.

B.7.2 Symmetry of the multinomial braids

Here we generalize the facts (13) and (18) that $\widetilde{b_i^{(n)}} = b_{n-i}^{(n)}$ and $\widetilde{f^{(n)}} = f^{(n)}$. For any $\eta = (\eta_1, \eta_2, \ldots, \eta_r)$, let $\tilde{\eta} = (\eta_r, \ldots, \eta_2, \eta_1)$.

Proposition. For any $\eta \in \mathcal{C}(n, r)$, $\widetilde{m^{(\eta)}} = m^{(\tilde{\eta})}$.

Proof. Consider the bijection $\mathcal{F}(n, r) \to \mathcal{F}(n, r)$, $f \to \tilde{f}$, where $\tilde{f}(i) = r + 1 - f(n + 1 - i)$. This clearly restricts to a bijection $\mathcal{S}(\eta) \to \mathcal{S}(\tilde{\eta})$, so it is enough to show that

$$\widetilde{s_f^{(n)}} = s_{\tilde{f}}^{(n)} \ \forall\ f \in \mathcal{S}(\eta)$$

to obtain the result.

We have that

$$\tilde{f}^{-1}(h) = n + 1 - f^{-1}(r + 1 - h),\ \forall\ h = 1, \ldots, r,$$

from where

$$\sigma_{\tilde{f}}(i) = n + 1 - \sigma_f(n + 1 - i) = \widetilde{\sigma_f}(i)\ \forall\ i = 1, \ldots, n,$$

and thus

$$\widetilde{s_f^{(n)}} = \widetilde{s_{\sigma_f}^{(n)}} \overset{(17)}{=} s_{\widetilde{\sigma_f}}^{(n)} = s_{\sigma_{\tilde{f}}}^{(n)} = s_{\tilde{f}}^{(n)}$$

as needed.

$\square$

B.7.3 Pascal's identity for multinomial braids

Let $\mathcal{C}^+(n,r)$ denote the set of *strict* compositions of n into r parts, i.e. those sequences $(\eta_1,\dots,\eta_r)$ such that $\eta_1+\dots+\eta_r=n$ *and* $\eta_i\in\mathbb{Z}^+\ \forall\ i=1,\dots,r$.

Pascal's identity (14) is actually a particular case of the following identity for multinomial braids.

Proposition. For any $\eta\in\mathcal{C}^+(n,r)$,

$$m^{(\eta_1,\eta_2,\dots,\eta_r)} = s^{(n)}(\eta_1,n)\cdot m^{(\eta_1-1,\eta_2,\dots,\eta_r)}{\otimes}1 + s^{(n)}(\eta_1+\eta_2,n)\cdot m^{(\eta_1,\eta_2-1,\dots,\eta_r)}{\otimes}1+\dots$$

$$\dots+s^{(n)}(\eta_1+\eta_2+\dots+\eta_r,n)\cdot m^{(\eta_1,\eta_2,\dots,\eta_r-1)}{\otimes}1. \quad (29)$$

Proof. Consider the bijection

$$\coprod_{i=1}^{r}\mathcal{S}(\eta_1,\dots,\eta_i-1,\dots,\eta_r)\to\mathcal{S}(\eta_1,\eta_2,\dots,\eta_r)$$

that sends $f\in\mathcal{S}(\eta_1,\dots,\eta_i-1,\dots,\eta_r)$ to $g\in\mathcal{S}(\eta_1,\eta_2,\dots,\eta_r)$ defined by

$$g(j)=\begin{cases} f(j) & \text{if } j\in\{1,2,\dots,n-1\},\\ i & \text{if } j=n.\end{cases}$$

Clearly,

$$e_j(g)=\begin{cases} e_j(f) & \text{if } j\in\{1,2,\dots,n-1\},\\ \eta_1+\dots+\eta_i & \text{if } j=n.\end{cases}$$

Hence $s_g^{(n)}=s^{(n)}(\eta_1+\eta_2+\dots+\eta_i,n)\cdot s_{n-1}^{(f)}{\otimes}1$. The result follows by summing over all such $f's$. $\square$

B.7.4 Multinomials in terms of binomials and factorials

In this section we relate the multinomial braids to the binomials and factorials, obtaining identities that generalize (21) and (22).

Proposition. Let $(\eta_1, \ldots, \eta_r) \in \mathcal{C}(n,r)$, $s \leq r$, and $n_1 = \eta_1 + \ldots + \eta_s$, $n_2 = \eta_{s+1} + \ldots + \eta_r$. Then

$$m^{(\eta_1,\ldots,\eta_r)} = m^{(\eta_1,\ldots,\eta_s)} \otimes m^{(\eta_{s+1},\ldots,\eta_r)} \cdot m^{(n_1,n_2)} \ . \tag{30}$$

Proof. Consider the bijection

$$\begin{aligned} \mathcal{S}(\eta_1, \ldots, \eta_r) &\to \mathcal{S}(\eta_1, \ldots, \eta_s) \times \mathcal{S}(\eta_{s+1}, \ldots, \eta_r) \times \mathcal{S}(n_1, n_2) \\ f &\mapsto (f_1, f_2, I) \end{aligned}$$

defined as follows:

$$\begin{aligned} I &= \{j \in \{1, \ldots, n\} \ / \ f(j) \leq s\} = \{j_1 < j_2 < \ldots < j_{n_1}\} \in \mathcal{S}_{n_1}(n), \\ I^c &= \{k \in \{1, \ldots, n\} \ / \ f(k) > s\} = \{k_1 < k_2 < \ldots < k_{n_2}\} \in \mathcal{S}_{n_2}(n), \\ f_1 &= \left(\begin{smallmatrix} 1 & 2 & \cdots & n_1 \\ f(j_1) & f(j_2) & \cdots & f(j_{n_1}) \end{smallmatrix}\right) \in \mathcal{S}(\eta_1, \ldots, \eta_s), \\ f_2 &= \left(\begin{smallmatrix} 1 & 2 & \cdots & n_2 \\ f(k_1)-s & f(k_2)-s & \cdots & f(k_{n_2})-s \end{smallmatrix}\right) \in \mathcal{S}(\eta_{s+1}, \ldots, \eta_r). \end{aligned}$$

(Informally, $f_1 = f|_I$, $f_2 = f|_{I^c}$.)

It is enough to show that

$$s_f^{(n)} = s_{n_1}^{(f_1)} \otimes s_{n_2}^{(f_2)} \cdot s_I^{(n)} \ .$$

We start by noting that for any $j \in \{1, \ldots, n\}$,

$$\begin{aligned} e_j(f) &= \#\{h \in \{1, \ldots, n\} \ / \ h \leq j \text{ and } f(h) \leq f(j)\} \\ &= \#\{h \in I \ / \ h \leq j \text{ and } f(h) \leq f(j)\} + \#\{h \in I^c \ / \ h \leq j \text{ and } f(h) \leq f(j)\}. \end{aligned}$$

Thus, if $j = j_i \in I$,

$$e_j(f) = \#\{h \in I \ / \ h \leq j \text{ and } f(h) \leq f(j)\} = e_i(f_1), \tag{*}$$

while if $j = k_i \in I^c$,

$$\begin{aligned} e_j(f) &= \#\{h \in I \ / \ h \leq j\} + \#\{h \in I^c \ / \ h \leq j \text{ and } f(h) \leq f(j)\} \\ &= \#\{1, 2, \ldots, k_i\} - \#\{h \in I^c \ / \ h \leq k_i\} + e_i(f_2) \\ &= k_i - i + e_i(f_2). \end{aligned} \tag{**}$$

Now,

$$\begin{aligned} & s^{(f_1)}_{n_1} \otimes s^{(f_2)}_{n_2} \cdot s^{(n)}_I = 1^{(n_1)} \otimes s^{(f_2)}_{n_2} \cdot s^{(f_1)}_{n_1} \otimes 1^{(n_2)} \cdot s^{(n)}_I = \\ & \overset{(2)}{=} 1^{(n_1)} \otimes s^{(f_2)}_{n_2} \cdot s^{(n)}(e_{n_1}(f_1), n_1) \cdot \ldots \cdot s^{(n)}(e_2(f_1), 2) \cdot s^{(n)}(e_1(f_1), 1) \cdot \\ & \quad \cdot s^{(n)}(n_1, j_{n_1}) \cdot \ldots \cdot s^{(n)}(2, j_2) \cdot s^{(n)}(1, j_1) \\ & \overset{(A1),\,(1)}{=} 1^{(n_1)} \otimes s^{(f_2)}_{n_2} \cdot s^{(n)}(e_{n_1}(f_1), j_{n_1}) \cdot \ldots \cdot s^{(n)}(e_2(f_1), j_2) \cdot s^{(n)}(e_1(f_1), j_1) \\ & \overset{(3)}{=} s^{(n)}(n_1 + e_{n_2}(f_2), n_1 + n_2) \cdot s^{(n)}(n_1 + e_2(f_2), n_1 + 2) \cdot s^{(n)}(n_1 + e_1(f_2), n_1 + 1) \cdot \\ & \quad \cdot s^{(n)}(e_{n_1}(f_1), j_{n_1}) \cdot \ldots \cdot s^{(n)}(e_2(f_1), j_2) \cdot s^{(n)}(e_1(f_1), j_1). \end{aligned}$$

At this point there are two cases to distinguish, according to whether $k_1 = n_1 + 1$ or $k_1 \leq n_1$ (notice that necessarily $k_1 \leq n_1 + 1$, since k_1 is the first element of I^c).

If $k_1 = n_1 + 1$ then necessarily $j_i = i$ and $k_i = n_1 + i \ \forall \ i$, so

$$\begin{aligned} & s^{(n)}(n_1 + e_i(f_2), n_1 + i) = s^{(n)}(k_i - i + e_i(f_2), n_1 + i) \overset{(**)}{=} \\ & \qquad = s^{(n)}(e_{k_i}(f), n_1 + i) = s^{(n)}(e_{n_1+i}(f), n_1 + i) \end{aligned}$$

and

$$s^{(n)}(e_i(f_1), j_i) \overset{(*)}{=} s^{(n)}(e_{j_i}(f_1), j_i) = s^{(n)}(e_i(f), i) \ .$$

Thus, in this case, all the factors in the above expression for $s^{(f_1)}_{n_1} \otimes s^{(f_2)}_{n_2} \cdot s^{(n)}_I$ are already in the "right order":

$$\begin{aligned} & s^{(f_1)}_{n_1} \otimes s^{(f_2)}_{n_2} \cdot s^{(n)}_I = \\ & \qquad = s^{(n)}(e_{n_1+n_2}(f), n_1+n_2) \cdot \ldots \cdot s^{(n)}(e_{n_1+1}(f), n_1+1) \cdot \\ & \qquad\qquad \cdot s^{(n)}(e_{n_1}(f), n_1) \cdot \ldots \cdot s^{(n)}(e_1(f), 1) = s^{(n)}_f, \end{aligned}$$

as needed.

The other case occurs when $k_1 \leq n_1$. In this case j_{k_1} is well-defined. We will move $s^{(n)}(n_1 + e_1(f_2), n_1 + 1)$ to its right past the factors $x_i := s^{(n)}(e_i(f_1), j_i)$ from $i = n_1$ down to $i = k_1$, using (5). We illustrate this process as follows:

$$\begin{aligned} & s^{(n)}(n_1 + e_1(f_2), n_1 + 1) \xrightarrow{\text{past } x_{n_1}} s^{(n)}(n_1 - 1 + e_1(f_2), n_1) \xrightarrow{\text{past } x_{n_1-1}} \ldots \xrightarrow{\text{past } x_{i+1}} \\ & \quad s^{(n)}(i + e_1(f_2), i+1) \xrightarrow{\text{past } x_i} s^{(n)}(i - 1 + e_1(f_2), i) \xrightarrow{\text{past } x_{i-1}} \ldots \\ & \qquad \ldots \xrightarrow{\text{past } x_{k_1}} s^{(n)}(k_1 - 1 + e_1(f_2), k_1) \overset{(**)}{=} s^{(n)}(e_{k_1}(f), k_1). \end{aligned}$$

Before proceeding, we must check that the hypothesis of (5) hold, in order to validate this commutation. In this situation those hypothesis are

$$e_i(f_1) \leq i - 1 + e_1(f_2) \text{ and } i \leq j_i - 1, \ \forall \ i \in \{k_1, \ldots, n_1\}.$$

The first inequality holds because, for any f and g, $e_i(f) \leq i$ and $e_1(g) \geq 1$. And the second one does too, for if not, we would have that $j_i \leq i$ and hence

$\{j_1, j_2, \ldots, j_i\} = \{1, 2, \ldots, i\}$. But since $k_1 \leq i$, this would imply that $k_1 \in I$, a contradiction. Thus the commutation process described above is valid.

Returning to the main argument, we next proceed similarly with the remaining factors $s^{(n)}(n_1 + e_2(f_2), n_1 + 2), \ldots, s^{(n)}(n_1 + e_{n_2}(f_2), n_1 + n_2)$, moving them to the right until they become $s^{(n)}(e_{k_2}(f), k_2)$, … ,$s^{(n)}(e_{k_{n_2}}(f), k_{n_2})$. After this has been done we are left with all the factors in the "right order":

$$s^{(f_1)}_{n_1} \otimes s^{(f_2)}_{n_2} \cdot s^{(n)}_I =$$

$$= s^{(n)}(e_{n_1+n_2}(f), n_1 + n_2) \cdot \ldots \cdot s^{(n)}(e_{n_1+1}(f), n_1 + 1) \cdot$$

$$\cdot s^{(n)}(e_{n_1}(f), n_1) \cdot \ldots \cdot s^{(n)}(e_1(f), 1) = s^{(n)}_f .$$

This completes the proof. □

From (30) we can easily deduce expressions for the multinomial braids in terms of binomials or factorials, that generalize well-known q-formulas.

Corollary.

$$m^{(\eta_1,\ldots,\eta_r)} = \tag{31}$$

$$= 1^{(\eta_1+\ldots+\eta_{r-1})} \otimes b^{(\eta_r)}_{\eta_r} \cdot 1^{(\eta_1+\ldots+\eta_{r-2})} \otimes b^{(\eta_{r-1}+\eta_r)}_{\eta_{r-1}} \cdot \ldots \cdot 1^{(\eta_1)} \otimes b^{(\eta_2+\ldots+\eta_r)}_{\eta_2} \cdot b^{(\eta_1+\ldots+\eta_r)}_{\eta_1}$$

$$m^{(\eta_1,\ldots,\eta_r)} = \tag{32}$$

$$= b^{(\eta_1)}_0 \otimes 1^{(\eta_2+\ldots+\eta_r)} \cdot b^{(\eta_1+\eta_2)}_{\eta_1} \otimes 1^{(\eta_3+\ldots+\eta_r)} \cdot \ldots \cdot b^{(\eta_1+\ldots+\eta_{r-1})}_{\eta_1+\ldots+\eta_{r-2}} \otimes 1^{(\eta_r)} \cdot b^{(\eta_1+\ldots+\eta_r)}_{\eta_1+\ldots+\eta_{r-1}}$$

$$f^{(\eta_1)} \otimes \ldots \otimes f^{(\eta_r)} \cdot m^{(\eta_1,\ldots,\eta_r)} = f^{(\eta_1+\ldots+\eta_r)} \tag{33}$$

Proof. Choosing $s = 1$ in equation (30) we get

$$m^{(\eta_1,\ldots,\eta_r)} = 1^{(\eta_1)} \otimes m^{(\eta_2,\ldots,\eta_r)} \cdot b^{(\eta_1+\ldots+\eta_r)}_{\eta_1} .$$

From here (31) follows immediately by induction on r.

Similarly, (32) follows by induction on r from

$$m^{(\eta_1,\dots,\eta_r)} = m^{(\eta_1,\dots,\eta_{r-1})} \otimes 1^{(\eta_r)} \cdot b^{(\eta_1+\dots+\eta_r)}_{\eta_1+\dots+\eta_{r-1}},$$

which is the case $s = r-1$ of (30).

The remaining identity can also be obtained by induction on r, as follows:

$$f^{(\eta_1)} \otimes \dots \otimes f^{(\eta_r)} \cdot m^{(\eta_1,\dots,\eta_r)} \overset{(30)}{=}$$
$$= \left[f^{(\eta_1)} \otimes \dots \otimes f^{(\eta_s)} \cdot m^{(\eta_1,\dots,\eta_s)}\right] \otimes \left[f^{(\eta_{s+1})} \otimes \dots \otimes f^{(\eta_r)} \cdot m^{(\eta_{s+1},\dots,\eta_r)}\right] \cdot m^{(n_1,n_2)} \overset{\text{(ind.hyp.)}}{=}$$
$$= f^{(n_1)} \otimes f^{(n_2)} \cdot m^{(n_1,n_2)} \overset{(22)}{=} f^{(n)}.$$

□

B.7.5 Witt's identity

The following identity for q-multinomials is a particular case of an identity that holds for all finite reflection groups, sometimes known as Witt's identity:

$$\sum_{r=0}^{n} (-1)^r \sum_{\eta \in \mathcal{C}^+(n,r)} \left[\begin{smallmatrix} n \\ \eta \end{smallmatrix}\right] = (-1)^n q^{\binom{n}{2}}$$

(this is [H, proposition 1.11] for the case of the reflection group S_n).

Recall that $\mathcal{C}^+(n,r)$ denotes the set of strict compositions of n into r parts. We should agree that $\mathcal{C}^+(n,0) = \begin{cases} \emptyset & \text{if } n > 0, \\ \{0\} & \text{if } n = 0 \end{cases}$, and that $m^{(0)} = 1 \in B_0$.

Witt's identity can be generalized to braids as follows.

Proposition. For every $n \geq 0$,

$$\sum_{r=0}^{n}(-1)^r \sum_{\eta\in\mathcal{C}^+(n,r)} m^{(\eta)} = \mu^{(n)}\ . \tag{34}$$

Proof. We do induction on n. For $n = 0$ the statement is obvious. Assume $n \geq 1$. Consider the decomposition

$$\coprod_{k=0}^{n-1}\mathcal{C}^+(k, r-1) \xrightarrow{\cong} \mathcal{C}^+(n,r),\ (\eta_1, \dots, \eta_{r-1}) \mapsto (\eta_1, \dots, \eta_{r-1}, n-k).$$

Recall that, by (32), for any $\eta \in \mathcal{C}^+(k, r-1)$ we have

$$m^{(\eta, n-k)} = m^{(\eta)}{\otimes}1^{(n-k)} \cdot b_k^{(n)}\ .$$

Hence

$$\sum_{\eta\in\mathcal{C}^+(n,r)} m^{(\eta)} = \sum_{k=0}^{n-1}\sum_{\eta\in\mathcal{C}^+(k,r-1)} m^{(\eta,n-k)} = \sum_{k=0}^{n-1}\sum_{\eta\in\mathcal{C}^+(k,r-1)} m^{(\eta)}{\otimes}1^{(n-k)} \cdot b_k^{(n)}\ . \tag{*}$$

Thus,

$$\begin{aligned}
&\sum_{r=0}^{n}(-1)^r \sum_{\eta\in\mathcal{C}^+(n,r)} m^{(\eta)} \overset{(n\geq 0)}{=} \\
&= \sum_{r=1}^{n}(-1)^r \sum_{\eta\in\mathcal{C}^+(n,r)} m^{(\eta)} \overset{(*)}{=} \sum_{k=0}^{n-1}\left[\sum_{r=1}^{n}(-1)^r \sum_{\eta\in\mathcal{C}^+(k,r-1)} m^{(\eta)}\right]{\otimes}1^{(n-k)} \cdot b_k^{(n)} \\
&= -\sum_{k=0}^{n-1}\left[\sum_{s=0}^{n-1}(-1)^s \sum_{\eta\in\mathcal{C}^+(k,r-1)} m^{(\eta)}\right]{\otimes}1^{(n-k)} \cdot b_k^{(n)} \overset{\text{(ind.hyp.)}}{=} \ -\sum_{k=0}^{n-1}\mu^{(k)}{\otimes}1^{(n-k)} \cdot b_k^{(n)} \\
&\overset{(26)}{=} \mu^{(n)}\ .
\end{aligned}$$

$\square$

B.8 Galois, Fibonacci and Catalan braids

The q-numbers

$$G_n = \sum_{k=0}^{n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$$

are studied in [GR2], where they are called the Galois numbers. They satisfy the following recurrence, that when $q = 1$ simply says that $G_n = 2^n$:

$$G_{n+1} = 2G_n + (q^n - 1)G_{n-1} \ .$$

One may define Galois braids $G^{(n)} \in kB_n$ as

$$G^{(n)} = \sum_{k=0}^{n} b_k^{(n)};$$

then one easily obtains the following generalization of the recurrence above:

$$\begin{aligned} G^{(n+1)} &= G^{(n)} \otimes 1 + 1 \otimes G^{(n)} + \\ &\quad + \sum_{k=0}^{n} s^{(n+1)}(1, k+1)^* s^{(n+1)}(k+1, n+1) \cdot 1 \otimes b_{n-1}^{(k-1)} \otimes 1 - 1 \otimes G^{(n-1)} \otimes 1 \ . \end{aligned}$$

Alternatively, one can define Galois braids $g^{(n)} \in kB_n$ as follows:

$$g^{(n)} = \sum_{k=0}^{n} c^{(k)} \otimes 1^{(n-k)} \cdot b_k^{(n)};$$

these braids satisfy the simpler formula:

$$g^{(n)} = [1 + s^{(n)}(1, n)] \cdot [1 + s^{(n)}(1, n-1)] \cdot \ldots \cdot [1 + s^{(n)}(1, 2)] \cdot [1 + s^{(n)}(1, 1)] \ ,$$

in fact, this is just the binomial theorem (23) at $\mathbf{x} = 1$, $\mathbf{y} = 0$, $\mathbf{z} = -1$.

These Galois braids $g^{(n)}$ specialize to Galois numbers

$$g_n = \sum_{k=0}^{n} q^{\binom{k}{2}} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$$

and the formula above becomes

$$g_n = (1 + q^{n-1}) \cdot (1 + q^{n-2}) \cdot \ldots \cdot (1 + q) \cdot (1 + 1) \ .$$

The Fibonacci numbers F_n count the number of subsets of $\{1, 2, \ldots, n\}$ without consecutive elements; one has $F_n = F_{n-1} + F_{n-2}$. It is easy to obtain q-versions of these numbers. More general braid analogs can be defined as follows. Let $\mathcal{F}(n, k)$ denote the set of subsets of $\{1, 2, \ldots, n\}$ with k elements no two of which are consecutive, and set

$$F_k^{(n)} = \sum_{I \in \mathcal{F}(n,k)} s_I^{(n)} \ \in \ kB_n.$$

As for the Galois braids, we have two options for defining the Fibonacci braids in terms of the $F_k^{(n)}$, according to whether we weight by the twistors $c^{(k)}$ or not. As before, weighting leads to simpler identities. So we define the Fibonacci braids $F^{(n)} \in kB_n$ as

$$F^{(n)} = \sum_{k=0}^{n} c^{(k)} \otimes 1^{(n-k)} \cdot F_k^{(n)} \ .$$

The same bijection considered in the proof of Pascal's identity (14) shows that

$$F_k^{(n)} = F_k^{(n-1)} \otimes 1 + s^{(n)}(k, n) \cdot F_{k-1}^{(n-2)} \otimes 1^{(2)};$$

from here it follows easily that

$$F^{(n)} = F^{(n-1)} \otimes 1 + s^{(n)}(1, n) \cdot F^{(n-2)} \otimes 1^{(2)} \ .$$

Thus these braids specialize to q-numbers F_n that satisfy

$$F_n = F_{n-1} + q^{n-1} F_{n-2} \ .$$

The Catalan numbers C_n count the number of subsets I of $\{1, 2, \ldots, 2n\}$ satisfying the following two conditions:

$$\#I = n \text{ and for every } j = 1, 2, \ldots, 2n,\ \#I \cap \{1, 2, \ldots, j\} \geq \#I^c \cap \{1, 2, \ldots, j\} \ .$$

Let $\mathcal{C}(n)$ denote the family of those subsets, and set

$$C^{(n)} = \sum_{I \in \mathcal{C}(n)} s_I^{(2n)} \ \in \ kB_{2n}.$$

It is easy to see from (*) in the proof of (13) that

$$C^{(n)} = \widetilde{C^{(n)}} \ .$$

Similarly, from (*) in the proof of (15) one deduces that

$$C^{(n+1)} = \sum_{k=0}^{n} 1^{(k+1)}{\otimes}\beta_{k+1,n-k}{\otimes}1^{(n-k)} \cdot 1{\otimes}C^{(k)}{\otimes}1{\otimes}C^{(n-k)} \ .$$

Thus these braids specialize to q-numbers C_n that satisfy

$$C_{n+1} = \sum_{k=0}^{n} q^{(k+1)(n-k)} C_k C_{n-k} \ .$$

These are the q-Catalan numbers of Carlitz and Riordan [CR].

B.9 Additional remarks

Further interesting combinatorial phenomena arises from the study of the behavior of the various braid analogs on higher dimensional representations X of the braid groups. In particular, the determinants of $b_i^{(n)}$ and $f^{(n)}$ on $X^{\otimes n}$ seem to factor in some rather remarkable ways, intimately related to the combinatorics of the braid arrangement $\mathcal{A}_{r-1} = \{H_{hk} \ / \ 1 \leq h < k \leq r\}$, where $H_{hk} = \{(x_1, \dots, x_r) \in \mathbb{R}^r \ / \ x_h = x_k\}$.

For instance, consider the representation constructed from a symmetric matrix $A = [a_{hk}]$ of size r as in section 9.8 of the main body of the thesis. Thus, B_n acts on $X^{\otimes n}$ $\forall$ $n \geq 0$, where X is a vector space with basis $\{x_1, \dots, x_r\}$. The subspace X_r of $X^{\otimes r}$ spanned by those tensors of the form $x_{\sigma(1)}{\otimes}x_{\sigma(2)}{\otimes}\dots{\otimes}x_{\sigma(r)}$, where σ runs over S_r, is invariant under the action of B_r. The matrix of $f^{(r)} : X_r \to X_r$ with respect to this basis turns out to be the same matrix that Varchenko associates to the weighted hyperplane arrangement $\mathcal{A}_{r-1}$ (weighted by the a_{hk}'s) [V]. A factorization formula for the determinant of the matrix of an arbitrary weighted real hyperplane arrangement is obtained in that work. For the special case of the braid arrangement, further factorization formulas seem to hold, not only for the determinant of the factorial braid, but also for the binomials, and on other invariant subspaces of $X^{\otimes n}$ as well.

In particular, on the subspace $X_{h,k}$ of $X^{\otimes(n+1)}$ spanned by $x_h{\otimes}x_k^{\otimes n}$ and its permutations, one can show that

$$\det\Big(b_1^{(n+1)}|_{X_{h,k}}\Big) = (q^{a_{hk}+a_{kh}}; q^{a_{kk}})_n [n]!_{q^{a_{kk}}},$$

where

$$(x;q)_n = (1-x)(1-qx)(1-q^2x)\cdots(1-q^{n-1}x) \ .$$

These questions will be the subject of future work.

Bibliography

[A] M.Aguiar, *Braids, q-binomials and quantum groups,* preprint, January 1997.

[Cha1] S.U.Chase, *Infinitesimal group scheme actions on finite field extensions,* Amer. J. Math 98 (1976) 34-79.

[Cha2] S.U.Chase, *A generalization of the ring of triangular matrices,* Nagoya Math. J. 18 (1961) 13-25.

[CPP] A.Carboni, S.Pedicchio, N.Pirovano, *Internal Graphs and Internal Categories in Malcev Categories,* CMS Conference Proceedings v13 (1992) 97-109.

[CR] L.Carlitz, J.Riordan *Two element lattice permutation numbers and their q-generalization,* Duke J. Math. 31 (1964), 371-388.

[CR1] C.Cibils, M.Rosso, *Algèbres des chamins quantiques,* Adv. Math. 125 (1997), 171-199.

[CR2] C.Cibils, M.Rosso, *Hopf bimodules are modules,* preprint (1996).

[Deh] P.Dehornoy, *From large cardinals to braids via ditributive algebra,* in Jour. Knot Theory 4 n1 (1995), 33-79.

[Del] P.Deligne, *Cátegories Tannakiennes,* in The Grothendieck Festchrift, II, Progress in Math, Birkhäuser (1990) 11-195.

[Doi1] Y.Doi, *Homological coalgebra,* J. Math. Soc. Japan 33 (1981) 31-50.

[Doi2] Y.Doi, *Hopf extensions of algebras and Maschke-type theorems,* Israel J. Math.72 (1990) 99-108.

[Dri1] V.G.Drinfeld, *Quantum Groups,* in Proc. Int. Cong. Math. (Berkeley, 1986), 798-820.

[Dri2] V.G.Drinfeld, *On some unsolved problems in quantum groups,* Lecture Notes in Math 1510 (1992), 1-8.

[FRT] L.Fadeev, N.Reshetikhin and L.Takhatjan, *Quantization of Lie groups and Lie algebras,* Leningrad Math. J. 1 (1990), 193ff.

[Gab] P.Gabriel, *Représentations indécomposables,* Séminaire Bourbaki 444 (1973/74), 143-168.

[GR1] J.Goldman, G.C.Rota, *On the foundations of Combinatorial Theory IV. Finite Vector Spaces and Eulerian Generating Functions,* Studies in Applied Mathematics vXLIX n3 (1970), 239-258.

[GR2] J.Goldman, G.C.Rota, *The number of subspaces of a vector space,* Recent Progress in Combinatorics, Academic Press (1969), 75-83.

[Gra] G.Grätzer, *General Lattice Theory,* Pure and Applied Mathematics 75, Academic Press (1978).

[HM] P. Higgins, K.Mackenzie, *Duality for base-changing morphisms of vector bundles, modules, Lie algebroids and Poisson structures,* Math. Proc. Cambridge Philos. Soc. 114 (1993), no 3, 471-488.

[H] J.E.Humphreys, *Reflection Groups and Coxeter Groups,* Cambridge Stud. Adv. Math. 29, Cambridge U. Press (1990).

[Jan] J.C.Jantzen, *Lectures on Quantum Groups,* GSM 6, American Mathematical Society (1996).

[Jim] M.Jimbo, *A q-difference analogue of* $U(g)$ *and the Yang-Baxter equation,* Lett. Math. Phys. 10 (1985), 63-69.

[Joh] P.T.Johnstone, *Topos Theory,* LMS Monographs 10 (1977).

[Kac] V.G..Kac, *Infinite dimensional Lie algebras,* Cambridge U. Press (1990).

[K] C.Kassel, *Quantum Groups,* GTM 155, Springer-Verlag (1995).

[KS] G.M.Kelly, R.Street *Review of the elements of 2-categories,* Lect. Notes Math. 420, Springer-Verlag (1974) 75-103.

[KV] M.M.Kapranov, V.A.Voevodsky *2-categories and Zamolodchikov Tetrahedra Equations,* Proc. Symp. in Pure Math. 56 (1995) 177-259.

[Lod1] J.L.Loday, *Spaces with finitely many non-trivial homotopy groups,* JPAA 24 (1982) 179-202.

[Lod2] J.L.Loday, *Cyclic Homology,* Grundlehren Math. Wiss. 301, Springer-Verlag (1992).

[Lus] G.Lusztig, *Introduction to Quantum Groups,* Progr. Math. 110, Birkhäuser (1993).

[LW] J.-H.Lu and A.Weinstein, *Poisson-Lie groups, dressing transformations and Bruhat decompositions,* J. Differential Geom. 31 (1990), 501ff.

[Mac] K.Mackenzie, *Lie Grupoids and Lie Algebroids in Differential Geometry,* LMS 124, Cambridge U Press (1987).

[Maj] S.Majid, *Foundations of Quantum Group Theory,* Cambridge U Press (1995).

[Mal] G.Maltsiniotis, *Groupoïdes quantiques,* C.R. Acad. Sci. Paris Sér. I Math. 314 (1992) 249-252.

[Mit] B.Mitchell, *Rings with several objects,* Adv. Math. 8 (1972) 1-161.

[Mon] S.Montgomery, *Hopf Algebras and Their Actions on Rings,* CBMS 82 (1993).

[ML] S.Mac Lane, *Categories for the Working Mathematician,* GTM 5 (1971).

[N] K.Norrie, *Actions and automorphisms of crossed modules,* Bull. Soc. math. France 118 (1990) 129-146.

[P1] B.Pareigis, *Non-additive ring and module theory I,* Publ. Math. (Debrecen) 24 (1977) 189-204.

[P2] B.Pareigis, *A non-commutative non-cocommutative Hopf algebra in "nature",* J. Algebra 70 (1981) 356-374.

[Pas] D.S.Passman, *The algebraic structure of group rings,* Wiley&Sons (1977).

[Rad] D.Radford, *The structure of Hopf algebras with a projection ,* J. of Algebra 92 (1985) 322-347.

[Rav] D.Ravenel, *Complex cobordism and stable homotopy groups of spheres,* Academic Press (1986).

[Ros] M.Rosso, *Groupes quantiques et algèbres de battages quantiques,* C.R. Acad. Sci. Paris 320 (1995) 145-148.

[Rot] G.C.Rota, *On the foundations of Combinatorial Theory. I. Theory of Möbius functions,* Z. Wahrscheinlichkertstheorie und Verw. Gebiete 2, (1964) 340-368.

[Row] L.Rowen, *Ring Theory,* Academic Press (1988), Vol 2.

[Sch] P.Schauenburg, *A Characterization of the Borel-like Subalgebras of Quantum Enveloping Algebras,* Comm. in Algebra 24 (1996), 2811-2823.

[Swe] M.Sweedler, *Hopf Algebras ,* W.A. Benjamin (1969).

[Taf] E.J.Taft, *The order of the antipode of a finite-dimensional Hopf algebra,* Proc. Nat. Acad. Sci. USA 68 (1971) 2631-2633.

[T1] M.Takeuchi, *Formal Schemes Over Fields,* Comm. Algebra 5 (1977) 1483-1528.

[T2] M.Takeuchi, *Morita theorems for categories of comodules,* J. Fac. Sci. Univ. Tokyo 24 (1977) 629-644.

[V] A.Varchenko, *Bilinear form of a real configuration of hyperplanes,* Adv. Math. 97 (1993), 110-144.

[W] J.H.C.Whitehead, *On operators in relative homotopy groups of spheres,* Ann. of Math. t49 (1948) 610-640.